

Linear Algebra 2: HW#3

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Winter 2023

due Friday, March 17, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (25 points). Let U and V be real vector spaces (i.e. vector spaces over \mathbb{R}), equipped with scalar products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$, respectively. Let $f : U \rightarrow V$ be a function that satisfies the property that

$$\langle f(\mathbf{u}), f(\mathbf{v}) \rangle_V = \langle \mathbf{u}, \mathbf{v} \rangle_U$$

for all $\mathbf{u}, \mathbf{v} \in U$. Prove that f is linear and one-to-one.

Hint: For linearity, consider

- $\langle f(\mathbf{u} + \mathbf{v}) - f(\mathbf{u}) - f(\mathbf{v}), f(\mathbf{u} + \mathbf{v}) - f(\mathbf{u}) - f(\mathbf{v}) \rangle_V$
- $\langle f(\alpha \mathbf{u}) - \alpha f(\mathbf{u}), f(\alpha \mathbf{u}) - \alpha f(\mathbf{u}) \rangle_V$

where $\mathbf{u}, \mathbf{v} \in U$ and $\alpha \in \mathbb{R}$.

Problem 2 (25 points). Find the least-squares solution(s) and the least-squares error of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 10^{-10} & 0 \\ 0 & 10^{-10} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 10^{-10} \\ 10^{-10} \end{bmatrix}$$

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent?

Problem 3 (25 points). Let $\langle \cdot, \cdot \rangle$ be a scalar product in \mathbb{R}^n (not necessarily the standard one). Let V be a real vector space (i.e. a vector space over \mathbb{R}), and let $f : V \rightarrow \mathbb{R}^n$ be a linear transformation. We define $\langle \cdot, \cdot \rangle_f : V \times V \rightarrow \mathbb{R}$ by setting

$$\langle \mathbf{u}, \mathbf{v} \rangle_f = \langle f(\mathbf{u}), f(\mathbf{v}) \rangle$$

for all $\mathbf{u}, \mathbf{v} \in V$. Prove that $\langle \cdot, \cdot \rangle_f$ is a scalar product on V if and only if f is one-to-one.

Problem 4 (25 points). Let $Q_1, Q_2 \in \mathbb{R}^{n \times m}$ be matrices with orthonormal columns,¹ and let $Q \in \mathbb{R}^{m \times m}$ be such that $Q_1 = Q_2 Q$. Prove that Q is an orthogonal matrix.

Hint: Consider the product $Q_1^T Q_1 = (Q_2 Q)^T (Q_2 Q)$.

¹Careful! If $m \neq n$, then Q_1 and Q_2 are **not** orthogonal. (Only square matrices can be orthogonal).