Linear Algebra 2:
HW#2

Irena Penev
Winter 2023

due Friday, March 3, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (25 points)**.** *Let V be a vector space over $\mathbb{R}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Prove that if vectors $\mathbf{x}, \mathbf{y} \in V$ satisfy*

$$||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2,$$

then they are orthogonal.

> ***Remark:*** *So, you are asked to prove the converse of the Pythagorean theorem for the case of real vector spaces (i.e. vector spaces over $\mathbb{R}$).*

Problem 2 (25 points)**.** *Consider the standard scalar product $\cdot$ on $\mathbb{C}^2$, and the induced norm $|| \cdot ||$. Find* ***non-orthogonal*** *vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^2$ that satisfy*

$$||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2.$$

> ***Remark:*** *This shows that the converse of the Pythagorean theorem need not hold for complex vector spaces (i.e. vector spaces over $\mathbb{C}$).*

Definition 1. *The* trace *of a square matrix* $A = [a_{i,j}]_{n\times n}$ *is* $trace(A) = \sum_{i=1}^{n} a_{i,i}$.[1]

Problem 3 (25 points)**.** *Consider the function* $\langle\cdot,\cdot\rangle : \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathbb{R}$ *given by*

$$\langle A, B\rangle \quad = \quad trace(A^T B)$$

for all $A, B \in \mathbb{R}^{n\times m}$*. Prove that* $\langle\cdot,\cdot\rangle$ *is a scalar product on* $\mathbb{R}^{n\times m}$*.*

Problem 4 (25 points)**.** *Let* V *be a vector space over* $\mathbb{R}$,[2] *equipped with a scalar product* $\langle\cdot,\cdot\rangle$ *and the induced norm* $||\cdot||$*. Let* $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ *be an orthogonal set of vectors in* V*, and let* $\alpha_1, \dots, \alpha_n \in \mathbb{R}$*. Prove that* $\{\alpha_1\mathbf{u}_1, \dots, \alpha_n\mathbf{u}_n\}$ *is an orthogonal set of vectors in* V*.*

> ***Remark:*** *This was used in the proof of correctness of the Gram-Schmidt orthogonalization process. You are asked to give a formal proof.*

[1] In other words, the *trace* of a square matrix is the sum of its entries on the main diagonal. For example, $\text{trace}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = 1 + 5 + 9 = 15$.

[2] Actually, this also works for $\mathbb{C}$, but for the sake of simplicity, you are only asked to prove it for $\mathbb{R}$.