Linear Algebra 2:
HW#1

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due Friday, February 24, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (30 points)**.** *Let $\langle \cdot, \cdot \rangle$ be a scalar product in a vector space V over $\mathbb{R}$ or $\mathbb{C}$, and let $|| \cdot ||$ be the norm in V induced by $\langle \cdot, \cdot \rangle$. Prove that*

$$||\mathbf{x} - \mathbf{y}||^2 + ||\mathbf{x} + \mathbf{y}||^2 \quad = \quad 2||\mathbf{x}||^2 + 2||\mathbf{y}||^2$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Problem 2 (20 points)**.** *The equality from Problem 1 holds for norms induced by scalar products, but it need not hold for other norms.*

(a) Prove that the equality from Problem 1 does **not** *hold for the Manhattan norm $|| \cdot ||_1$ on $\mathbb{R}^2$, that is, find vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that*

$$||\mathbf{x} - \mathbf{y}||_1^2 + ||\mathbf{x} + \mathbf{y}||_1^2 \quad \neq \quad 2||\mathbf{x}||_1^2 + 2||\mathbf{y}||_1^2.$$

(b) Prove that the equality from Problem 1 does **not** *hold for the Chebyshev distance $|| \cdot ||_\infty$ on $\mathbb{R}^2$, that is, find vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that*

$$||\mathbf{x} - \mathbf{y}||_\infty^2 + ||\mathbf{x} + \mathbf{y}||_\infty^2 \quad \neq \quad 2||\mathbf{x}||_\infty^2 + 2||\mathbf{y}||_\infty^2.$$

Problem 3 (50 points). *As usual,* $\mathbb{Z}_2^{2\times 2}$ *is the vector space (over* $\mathbb{Z}_2$*) of all* 2×2 *matrices with entries in* $\mathbb{Z}_2$*, and* $\mathbb{P}_{\mathbb{Z}_2}$ *is the vector space (over* $\mathbb{Z}_2$*) of all polynomials with coefficients in* $\mathbb{Z}_2$*.*

(a) *[20 points] Prove that there exists a unique linear transformation* $f : \mathbb{Z}_2^{2\times 2} \to \mathbb{P}_{\mathbb{Z}_2}$ *that satisfies all the following:*

- $f\Big(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\Big) = x^4 + x^3 + x^2 + x + 1;$
- $f\Big(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\Big) = x^3 + x^2 + x + 1;$
- $f\Big(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\Big) = x^2 + 1;$
- $f\Big(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\Big) = x^4 + x^2 + 1;$
- $f\Big(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Big) = x^5 + x^4 + x^3 + x + 1;$
- $f\Big(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\Big) = x^5 + x^3 + x^2 + x.$

(b) *[15 points] Find rank*(f)*, and determine whether* f *is one-to-one, where* f *is the linear transformation from part (a).*

(c) *[15 points] Find the formula for the linear transformation* $f : \mathbb{Z}_2^{2\times 2} \to \mathbb{P}_{\mathbb{Z}_2}$ *from part (a), that is, fill in the blank in the following:*

$$f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) \quad = \quad \underline{\hspace{4cm}} \qquad \forall a, b, c, d \in \mathbb{Z}_2.$$