

Linear Algebra 1

Lecture #1

Systems of linear equations

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- This lecture covers sections 1.1, 1.2, and 1.3 of the Lecture Notes (<https://iuuk.mff.cuni.cz/~ipenev/LALectureNotes.pdf>).
 - Subsection 1.3.7 is **optional reading** for the ambitious.

This lecture has three parts:

- ① An informal introduction to fields
- ② An introduction to matrices and vectors
- ③ Systems of linear equations and row reduction

① An informal introduction to fields

- A formal definition of a field will be given later in the course.
- For now, we give a few examples of fields:
 - the field $\mathbb{Q}$ of rational numbers;
 - the field $\mathbb{R}$ of real numbers;
 - the field $\mathbb{C}$ of complex numbers;
 - the field $\mathbb{Z}_p$, where p is a **prime** number.
 - If $n \in \mathbb{N}$ is not prime, then $\mathbb{Z}_n$ is **not** a field.

- Each field is equipped with two operations: addition and multiplication.
- These two operations are commutative and associative, and multiplication is distributive over addition:
 - $a + b = b + a$ and $ab = ba$;
 - $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$;
 - $a(b + c) = ab + ac$.
- Every field has an “additive identity” 0 and a “multiplicative identity” 1, which satisfy

$$a + 0 = 0 + a = a \quad \text{and} \quad a \cdot 1 = 1 \cdot a = a$$

for all elements a of the field.

- Every element a of a field has a corresponding “additive inverse,” denoted by $-a$, which is a number that we can add to a in order to obtain 0.
- For example:
 - the additive inverse of $\sqrt{17}$ in $\mathbb{R}$ is $-\sqrt{17}$, since $\sqrt{17} + (-\sqrt{17}) = 0$ in $\mathbb{R}$.
 - the additive inverse of $2 - i$ in $\mathbb{C}$ is $-2 + i$, since $(2 - i) + (-2 + i) = 0$ in $\mathbb{C}$;
 - the additive inverse of 3 in $\mathbb{Z}_5$ is 2 (and we write $-3 = 2$), since $3 + 2 = 0$ in $\mathbb{Z}_5$;
 - the additive inverse of 4 in $\mathbb{Z}_5$ is 1 (and we write $-4 = 1$), since $4 + 1 = 0$ in $\mathbb{Z}_5$;
 - the additive inverse of 2 in $\mathbb{Z}_3$ is 1 (and we write $-2 = 1$), since $2 + 1 = 0$ in $\mathbb{Z}_3$.

- Every **non-zero** element a of a field has a “multiplicative inverse,” denoted by a^{-1} , which is a number we can multiply a by in order to obtain 1.
- For example:
 - the multiplicative inverse of $\sqrt{17}$ in $\mathbb{R}$ is $\frac{1}{\sqrt{17}}$, because $\sqrt{17} \cdot \frac{1}{\sqrt{17}} = 1$ in $\mathbb{R}$;
 - the multiplicative inverse of $2 - i$ is $\frac{2}{5} + \frac{1}{5}i$, because $(2 - i)(\frac{2}{5} + \frac{1}{5}i) = 1$ in $\mathbb{C}$;
 - the multiplicative inverse of 3 in $\mathbb{Z}_5$ is 2 (and we write $3^{-1} = 2$), since $3 \cdot 2 = 1$ in $\mathbb{Z}_5$;
 - the multiplicative inverse of 4 in $\mathbb{Z}_5$ is 4 (and we write $4^{-1} = 4$), since $4 \cdot 4 = 1$ in $\mathbb{Z}_5$;
 - the multiplicative inverse of 2 in $\mathbb{Z}_3$ is 2 (and we write $2^{-1} = 2$), since $2 \cdot 2 = 1$ in $\mathbb{Z}_3$.

- **Remark:** When working over $\mathbb{Z}_p$ (for a prime number p), it is a good idea to first write out the addition and multiplication tables for $\mathbb{Z}_p$, because this allows us to easily identify additive and multiplicative inverses: for a given $a \in \mathbb{Z}_p$, we simply read off from the tables what number we need to add to a to get zero, and (assuming $a \neq 0$) what number we need to multiply it by to get 1.
- **Warning:** The following are **not** fields: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_n$ (where n is a positive integer that is **not** prime).

For the remainder of chapter 1, you may assume that the field $\mathbb{F}$ in question is one of the following: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_p$ (where p is a prime number). However, everything that we prove in this chapter does in fact hold for general fields $\mathbb{F}$, not just the ones listed above.

2 An introduction to matrices and vectors

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 3 & 2 \\ 1 & -1 & -5 \\ -2 & 2 & 3 \end{bmatrix}$$

$\uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow$
 $2 \times 3 \qquad \qquad \qquad 3 \times 2 \qquad \qquad \qquad 3 \times 3$

- A *matrix* is a rectangular array of numbers (typically, elements of some field).
- An $n \times m$ *matrix* (read “ n by m matrix”) is a matrix with n rows and m columns.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 3 & 2 \\ 1 & -1 & -5 \\ -2 & 2 & 3 \end{bmatrix}$$

$\uparrow$
 2×3

$\uparrow$
 3×2

$\uparrow$
 3×3

- A *square matrix* is one that has the same number of rows and columns.
 - So, C is a square matrix, but A and B are not square matrices.
- The *main diagonal* of a square matrix is the diagonal between the upper left corner and the bottom right corner.

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & -1 & -5 \\ -2 & 2 & 3 \end{bmatrix}$$

- The rows of a matrix are enumerated from top to bottom, whereas the columns are enumerated from left to right.

$$\begin{array}{rcccc}
 & 1 & 2 & \cdots & m \\
 & \downarrow & \downarrow & & \downarrow \\
 1 \rightarrow & \left[\begin{array}{cccc} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{array} \right] \\
 2 \rightarrow & & & & \\
 \vdots & & & & \\
 n \rightarrow & & & &
 \end{array}$$

- The i, j -th entry of a matrix is the entry that appears in the i -th row (from the top) and j -th column (from the left) in the matrix.

- A matrix A can be specified as follows:

$$A = \left[a_{i,j} \right]_{n \times m}.$$

- This notation indicates that the matrix A is of size $n \times m$ (i.e. has n rows and m columns), and the i, j -th entry (i.e. the entry in the i -th row and j -th column) is $a_{i,j}$.
- So, if $A = \left[a_{i,j} \right]_{n \times m}$, then we have that

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{bmatrix}.$$

- A *zero matrix* is a matrix all of whose entries are 0 (where the 0 comes from the field that we are working with).
- The zero matrix of size $n \times m$ is denoted by $O_{n \times m}$.
 - For example,

$$O_{2 \times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- A *non-zero matrix* is a matrix that has at least one non-zero entry.
- **Notation:** If $\mathbb{F}$ is a field, then the set of all $n \times m$ matrices with entries in $\mathbb{F}$ is denoted by $\mathbb{F}^{n \times m}$.
- **Terminology:** A *real matrix* is a matrix whose entries are real numbers, whereas a *complex matrix* is a matrix whose entries are complex numbers.

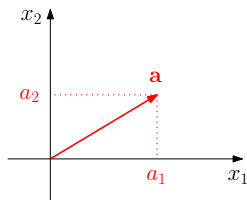
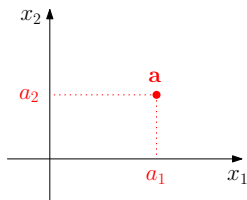
$$\mathbf{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -13 \\ 0 \\ 0 \\ \pi \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

- A *column vector*, or simply *vector*, is a matrix with just one column.
- Vectors are typically denoted by bold letters (e.g. $\mathbf{a}$, $\mathbf{u}$, $\mathbf{x}$) or by letters with an arrow on top (e.g. $\vec{a}$, $\vec{u}$, $\vec{x}$).

- The zero vector (i.e. vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$) is denoted by $\mathbf{0}$ or $\vec{0}$.
 - The number of entries in a zero vector should either be made explicit or be clear from context.
- A *non-zero vector* is a vector that has at least one non-zero entry.
- **Notation:** If $\mathbb{F}$ is a field, then the set of all (column) vectors with n entries, all of them in $\mathbb{F}$, is denoted by $\mathbb{F}^n$.
 - Thus, $\mathbb{F}^n = \mathbb{F}^{n \times 1}$.

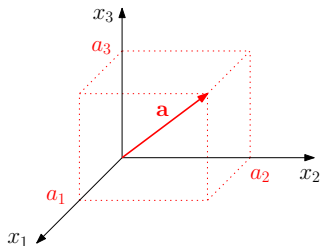
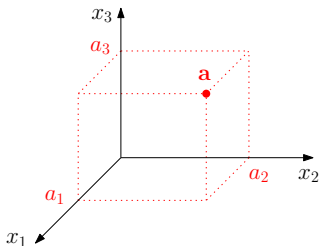
- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have a geometric interpretation.

- A vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ in $\mathbb{R}^2$ can be represented in the two-dimensional Euclidean space either as a point or as a line segment with an arrow starting at the origin.



- The zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is simply the origin.

- A vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ in $\mathbb{R}^3$ has a similar geometric interpretation in the three-dimensional Euclidean space.



- Once again, the zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is simply the origin.
- Vectors in $\mathbb{R}^n$ for $n \geq 4$ are higher-dimensional analogs of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

- A *row vector* is a matrix with only one row.
- For example, the following are row vectors:
 - $\mathbf{a} = \begin{bmatrix} 1 & -3 \end{bmatrix}$;
 - $\mathbf{b} = \begin{bmatrix} -13 & 0 & 0 & \pi \end{bmatrix}$;
 - $\mathbf{c} = \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \end{bmatrix}$.
- The set of all row vectors with n entries, all of them in some field $\mathbb{F}$, is denoted by $\mathbb{F}^{1 \times n}$ (i.e. exactly the same way as the set of all $1 \times n$ matrices with entries in $\mathbb{F}$).

- The columns of a matrix can be seen as (column) vectors, and matrices can be specified in terms of their columns.
- When we specify a matrix $A \in \mathbb{F}^{n \times m}$ (where $\mathbb{F}$ is some field) in the form

$$A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix},$$

we mean that $\mathbf{a}_1, \dots, \mathbf{a}_m$ are the columns of A (appearing in that order from left to right in the matrix A), and moreover, $\mathbf{a}_1, \dots, \mathbf{a}_m$ are vectors in $\mathbb{F}^n$.

- For example, if $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$, where

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

$$\text{then } A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 4 \end{bmatrix}.$$

- Similarly, the rows of a matrix can be seen as row vectors, and matrices can be specified in terms of their rows.
- When we specify a matrix $A \in \mathbb{F}^{n \times m}$ (where $\mathbb{F}$ is some field) in the form

$$A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \end{bmatrix},$$

we mean that $\mathbf{r}_1, \dots, \mathbf{r}_n$ are the rows of A (appearing in that order from top to bottom in the matrix A), and moreover, $\mathbf{r}_1, \dots, \mathbf{r}_n$ are row vectors in $\mathbb{F}^{1 \times m}$.

- For example, if $A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$, where

$$\mathbf{r}_1 = \begin{bmatrix} 1 & 2 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{bmatrix} 3 & 4 & 4 & 3 \end{bmatrix},$$

$$\text{then } A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & 4 & 3 \end{bmatrix}.$$

3 Systems of linear equations and row reduction

- A *linear equation* in the variables $x_1, \dots, x_m$ is an equation that can be written in the form

$$a_1x_1 + \cdots + a_mx_m = b,$$

where b and the coefficients $a_1, \dots, a_n$ are elements of some field $\mathbb{F}$.

- For example, $x_1 - 3(x_2 - x_1) = 7x_3 - 4$, with coefficients understood to be in $\mathbb{R}$, is a linear equation because it can be algebraically rearranged to have the form $4x_1 - 3x_2 - 7x_3 = -4$, which is obviously a linear equation.
- On the other hand, equations $x_1^3 + x_2 = 17$ and $x_1 - \sqrt{x_2} = 5$ are **not** linear because of x_1^3 and $\sqrt{x_2}$.

- A *system of linear equations*, or a *linear system*, is a collection of one or more linear equations involving the same variables, say $x_1, \dots, x_m$ (and with coefficients from the same field).
- For example, the following is a linear system (here, the coefficients are assumed to be in $\mathbb{R}$):

$$\begin{array}{ccccccccc} 2x_1 & + & 7x_2 & & & - & \pi x_4 & = & -\sqrt{3} \\ & & -3x_2 & + & 17x_3 & - & 3x_4 & = & 2 \\ x_1 & + & x_2 & - & 2x_3 & + & 7x_4 & = & \frac{11}{2} \end{array}$$

- **Remark:** Typographically, we normally arrange equations in our system so that the terms involving the same variable are below each other (i.e. visually in the same column).
- A *solution* of a linear system in variables $x_1, \dots, x_m$ is a list $s_1, \dots, s_m$ of numbers (from the same field as the coefficients of the system) s.t. each equation becomes a true statement when $s_1, \dots, s_m$ are substituted for $x_1, \dots, x_m$, respectively.

Example 1.3.1

Consider the linear system

$$\begin{array}{rcccccccl} x_1 & + & 2x_2 & - & x_3 & = & 9 \\ & & 2x_2 & + & 3x_3 & = & 16 \\ x_1 & + & x_2 & - & x_3 & = & 4 \end{array}$$

with coefficients in $\mathbb{R}$. Then

$$\begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 5 \\ x_3 & = & 2 \end{array}$$

is a solution of the system above.

Example 1.3.2

Consider the linear system

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & = & 0 \\ 2x_1 & + & x_2 & = & 1 \end{array}$$

with coefficients in $\mathbb{Z}_3$. Then

$$\begin{array}{rcrcrcrcl} x_1 & = & 1 \\ x_2 & = & 2 \end{array}$$

is a solution of the system above.

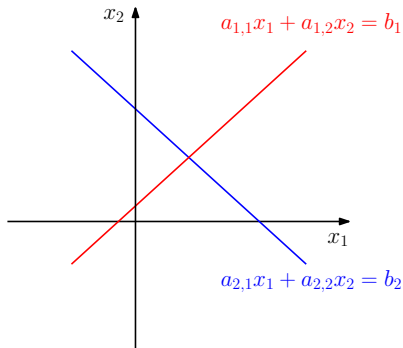
- The *set of solutions* or *solution set* of a linear system is the set of all solutions of that system.
- Our goal is to describe a procedure for finding the solution set of any linear system.
- A linear system may have no solutions, may have a unique solution (i.e. exactly one solution), or may have more than one solution.
- A system that has at least one solution is called *consistent*; a system that has no solutions is said to be *inconsistent*.

- Let us consider the geometry of linear systems with real coefficients.
- Consider the following system of two linear equations in two variables, with coefficients in $\mathbb{R}$.

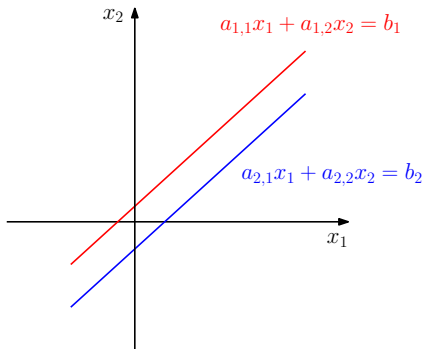
$$\begin{aligned}a_{1,1}x_1 + a_{1,2}x_2 &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 &= b_2\end{aligned}$$

- Let us assume that at least one of the coefficients $a_{1,1}, a_{1,2}$ is non-zero, and similarly, that at least one of the coefficients $a_{2,1}, a_{2,2}$ is non-zero.
- Then each of the two equations above defines a line in the plane.
- There are three possibilities for these two lines (next three slides):

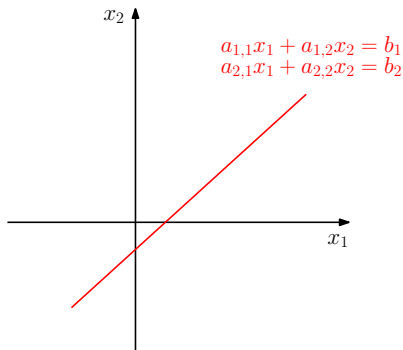
- The two lines may intersect in one point (in this case, the system has a unique solution, and in particular, it is consistent).



- The two lines may be distinct, parallel lines (in this case, the system has no solutions, i.e. it is inconsistent).



- The two lines may be identical (in this case, the system has infinitely many solutions, and in particular, the system is consistent).



- Note that the two lines may be identical even if the two equations are different. For instance, $x_1 + x_2 = 1$ and $2x_1 + 2x_2 = 2$ define the same line.

- Suppose now that we have a system of two linear equations in three variables (with coefficients in $\mathbb{R}$).

$$\begin{array}{ccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & a_{1,3}x_3 & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & a_{2,3}x_3 & = & b_2 \end{array}$$

- Let us assume that at least one of the coefficients $a_{1,1}, a_{1,2}, a_{1,3}$ is non-zero, and that at least one of the coefficients $a_{2,1}, a_{2,2}, a_{2,3}$ is non-zero.
- Then each of the two equations above defines a plane in the three-dimensional Euclidean space.
- Those two planes may **intersect in a line** (in which case the system has infinitely many solutions, and in particular, the system is consistent); or the two planes may **be distinct and parallel** (in which case, the system has no solutions, i.e. it is inconsistent); or the two planes may **be identical** (in which case the system has infinitely many solutions, and in particular, the system is consistent).

- Suppose we are given a system of n linear equations in m variables, as follows.

$$\begin{array}{ccccccc}
 a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & b_1 \\
 a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & b_2 \\
 & & & & & & \vdots & & \\
 a_{n,1}x_1 & + & a_{n,2}x_2 & + & \dots & + & a_{n,m}x_m & = & b_n
 \end{array}$$

- There are two matrices associated with this linear system: the “coefficient matrix” and the “augmented matrix.”
- The *coefficient matrix* of this system is the $n \times m$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{bmatrix}.$$

$$\begin{aligned}
 a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,m}x_m &= b_1 \\
 a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,m}x_m &= b_2 \\
 &\vdots \\
 a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,m}x_m &= b_n
 \end{aligned}$$

- The *augmented matrix* of our linear system is the $n \times (m + 1)$ matrix

$$\left[\begin{array}{cccc|c} A & \mathbf{b} \end{array} \right] = \left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \dots & a_{1,m} & b_1 \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} & b_n \end{array} \right],$$

where A is the coefficient matrix of the linear system, and

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

- linear system:

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & b_2 \\ & & & & & & & & \vdots \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \dots & + & a_{n,m}x_m & = & b_n \end{array}$$

- augmented matrix:

$$\left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \dots & a_{1,m} & b_1 \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} & b_n \end{array} \right]$$

- Obviously, a linear system is fully “encoded” by its augmented matrix.
- The vertical dotted line is optional, but serves as a helpful visual aid.

Example 1.3.3

Find the coefficient matrix and the augmented matrix of the linear system below (with coefficients understood to be in $\mathbb{R}$).

$$\begin{array}{rcrcrcrcrcl} 3x_1 & + & 2x_2 & + & 5x_3 & = & 7 \\ & & 3x_2 & - & x_3 & = & 0 \end{array}$$

Solution. The coefficient matrix of the linear system is

$$\begin{bmatrix} 3 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix},$$

whereas the augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 2 & 5 & 7 \\ 0 & 3 & -1 & 0 \end{array} \right].$$



- Two linear systems (with the same variables) are *equivalent* if they have exactly the same solution set.
- Now, suppose we are given a system of linear equations such as the one below (with coefficients understood to be in some field $\mathbb{F}$).

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & b_2 \\ & & & & & & \vdots & & \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \dots & + & a_{n,m}x_m & = & b_n \end{array}$$

- We would like to manipulate this system in a way that allows us to “read off” the solution set of the system.
- There are three basic ways that we can manipulate the system in a way that does not change the solution set (i.e. in a way that produces an equivalent linear system).

① Swap (interchange) two equations.

- For example, by swapping the first and third equation of the linear system on the left, we obtain the linear system on the right.

$$\begin{array}{rccccccccc} x_1 & + & 3x_2 & - & 2x_3 & = & -1 \\ \frac{1}{2}x_1 & & & + & 2x_3 & = & 0 \\ x_1 & + & x_2 & + & 2x_3 & = & 2 \end{array} \longrightarrow \begin{array}{rccccccccc} x_1 & + & x_2 & + & 2x_3 & = & 2 \\ \frac{1}{2}x_1 & & & + & 2x_3 & = & 0 \\ x_1 & + & 3x_2 & - & 2x_3 & = & -1 \end{array}$$

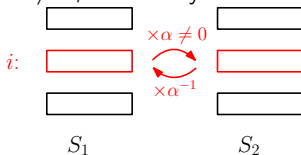
It is obvious that this operation does not alter the solution set.

② Multiply one equation by a **non-zero** scalar.

- For example, by multiplying the second equation of the linear system on the left by 2, we obtain the linear system on the right.

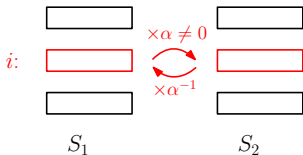
$$\begin{array}{rcccccccl}
 x_1 & + & x_2 & + & 2x_3 & = & 2 & & x_1 & + & x_2 & + & 2x_3 & = & 2 \\
 \frac{1}{2}x_1 & & & & + & 2x_3 & = & 0 & \longrightarrow & x_1 & & & + & 4x_3 & = & 0 \\
 x_1 & + & 3x_2 & - & 2x_3 & = & -1 & & x_1 & + & 3x_2 & - & 2x_3 & = & -1
 \end{array}$$

- Why does this not alter the solution set?
- Suppose we have multiplied the i -th equation of a linear system S_1 by some scalar $\alpha \neq 0$, and this yielded a linear system S_2 .



- Obviously, all solutions of the old system are still solutions of the new system.
- On the other hand, by multiplying the i -th equation of the new system by α^{-1} (the multiplicative inverse of α), we get the old system back.
- So, any solution of the new system is a solution of the old system as well.

- ② Multiply one equation by a **non-zero** scalar.



- **Warning:** Do **not** multiply an equation by 0, since that “kills” the equation!

- 3 Add a scalar multiple of one equation to another equation.
- For example, by adding (-1) times the second equation to the third equation of the linear system on the left, we obtain the linear system on the right.

$$\begin{array}{rclclcl}
 x_1 & + & x_2 & + & 2x_3 & = & 2 \\
 x_1 & & & + & 4x_3 & = & 0 \\
 x_1 & + & 3x_2 & - & 2x_3 & = & -1
 \end{array}
 \longrightarrow
 \begin{array}{rclclcl}
 x_1 & + & x_2 & + & 2x_3 & = & 2 \\
 x_1 & & & + & 4x_3 & = & 0 \\
 & & 3x_2 & - & 6x_3 & = & -1
 \end{array}$$

- Why does this not alter the solution set?
- Suppose we have added α times the i -th equation to the j -th equation (where $i \neq j$).

$$\begin{array}{ccc}
 R_i & \boxed{} & \boxed{} \quad R'_i = R_i \\
 & \boxed{} & \boxed{} \\
 R_j & \boxed{} & \boxed{} \quad R'_j = R_j + \alpha R_i \\
 S_1 & & S_2
 \end{array}$$

- Obviously, any solution of the old system is also a solution of the new system.
- On the other hand, if we start with the new system, then add $-\alpha$ times the i -th equation to the j -th equation, we get the old system back.
- So, any solution of the new system is a solution of the old system as well.

- Instead of manipulating systems linear systems in this way, we can manipulate their augmented matrices.
- There are three types of “elementary row operations” on matrices (with entries understood to be in some field $\mathbb{F}$):
 - 1 Swap (interchange) two rows.
 - We denote the operation of swapping rows i and j ($i \neq j$) by “ $R_i \leftrightarrow R_j$.”
 - For example, we can swap the first and third row of the matrix on the left to obtain the matrix on the right.

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ \frac{1}{2} & 0 & 2 & 0 \\ 1 & 1 & 2 & 2 \end{array} \right] \quad R_1 \leftrightarrow R_3 \quad \sim \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ \frac{1}{2} & 0 & 2 & 0 \\ 1 & 3 & -2 & -1 \end{array} \right]$$

② Multiply one row by a **non-zero** scalar.

- We denote the operation of multiplying row i by a scalar $\alpha \neq 0$ by " $R_i \rightarrow \alpha R_i$."
- For instance, we can multiply the second row of the matrix on the left by 2 to obtain the matrix on the right.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ \frac{1}{2} & 0 & 2 & 0 \\ 1 & 3 & -2 & -1 \end{array} \right] \xrightarrow[R_2 \rightarrow 2R_2]{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 0 & 4 & 0 \\ 1 & 3 & -2 & -1 \end{array} \right]$$

③ Add a scalar multiple of one row to another row.

- We denote the operation of adding scalar α times row i to row j ($i \neq j$) by " $R_j \rightarrow R_j + \alpha R_i$."
- For example, we can add (-1) times the second row to the third row of the matrix on the left to obtain the matrix on the right.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 0 & 4 & 0 \\ 1 & 3 & -2 & -1 \end{array} \right] \quad R_3 \rightarrow R_3 + \underset{\sim}{(-1)}R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 0 & 4 & 0 \\ 0 & 3 & -6 & -1 \end{array} \right]$$

Note: Instead of " $R_3 \rightarrow R_3 + (-1)R_2$," we could also have written (and we typically do write) just " $R_3 \rightarrow R_3 - R_2$."

- Elementary row operations:

- ① Swap (interchange) two rows.

- We denote the operation of swapping rows i and j ($i \neq j$) by " $R_i \leftrightarrow R_j$."

- ② Multiply one row by a **non-zero** scalar.

- We denote the operation of multiplying row i by a scalar $\alpha \neq 0$ by " $R_i \rightarrow \alpha R_i$."

- ③ Add a scalar multiple of one row to another row.

- We denote the operation of adding scalar α times row i to row j ($i \neq j$) by " $R_j \rightarrow R_j + \alpha R_i$."

- Importantly, all elementary row operations are reversible:

- ① we can undo (reverse) the operation of swapping two rows (" $R_i \leftrightarrow R_j$ ") by applying the same operation again;

- ② we can undo (reverse) the operation of multiplying row i by a scalar $\alpha \neq 0$ (" $R_i \rightarrow \alpha R_i$ ") by multiplying row i by α^{-1} (" $R_i \rightarrow \alpha^{-1} R_i$ ");

- ③ we can undo (reverse) the operation of adding scalar α times row i to another row j (" $R_j \rightarrow R_j + \alpha R_i$ ") by adding $-\alpha$ times row i to row j (" $R_j \rightarrow R_j - \alpha R_i$ ").

- **Remark:**

- Solving systems of linear equations is our primary motivation for introducing elementary row operations.
- However, we can, in principle, perform elementary row operations on **any** matrix (with entries in some field), even one that was not obtained as an augmented matrix of a linear system.
 - We will, indeed, do this at various points in this course.
- However, for now, it is useful to think of elementary row operations on matrices as a more compact way of performing the corresponding operations on linear systems.

- **Terminology/Notation:** If one matrix can be obtained from another via some sequence of elementary row operations, then the two matrices are said to be *row equivalent*. If matrices A and B are row equivalent, then we write $A \sim B$.
 - Note that any two row equivalent matrices are of the same size (i.e. have the same number of rows and the same number of columns), and their entries belong to the same field.
- **Remark:** Clearly, if two matrices with at least two columns (and with entries in some field $\mathbb{F}$) are row equivalent, then they encode equivalent linear systems (as augmented matrices).
 - A matrix that only has one column is not the augmented matrix of any linear system.
 - That said, according to our definition, two one-column matrices (i.e. two column vectors) can be row equivalent.

Proposition 1.3.5

Let $\mathbb{F}$ be a field. Then all the following hold:

- Ⓐ for all $A \in \mathbb{F}^{n \times m}$, $A \sim A$;
- Ⓑ for all $A, B \in \mathbb{F}^{n \times m}$, if $A \sim B$, then $B \sim A$;
- Ⓒ for all $A, B, C \in \mathbb{F}^{n \times m}$, if $A \sim B$ and $B \sim C$, then $A \sim C$.

Remark: Proposition 1.3.5 states that, for a field $\mathbb{F}$, row equivalence is an equivalence relation on the set $\mathbb{F}^{n \times m}$.

Proof. (a) Fix $A \in \mathbb{F}^{n \times m}$. By, for example, multiplying the first row of A by 1 (i.e. by applying the elementary row operation " $R_1 \rightarrow 1R_1$ "), we obtain the original matrix A ; so, $A \sim A$.

Proposition 1.3.5

Let $\mathbb{F}$ be a field. Then all the following hold:

- Ⓐ for all $A \in \mathbb{F}^{n \times m}$, $A \sim A$;
- Ⓑ for all $A, B \in \mathbb{F}^{n \times m}$, if $A \sim B$, then $B \sim A$;
- Ⓒ for all $A, B, C \in \mathbb{F}^{n \times m}$, if $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof (continued). (b) Fix $A, B \in \mathbb{F}^{n \times m}$, and assume that $A \sim B$. Then by applying some sequence $R_1, \dots, R_k$ of elementary row operations to A , we obtain the matrix B . But we know that elementary row operations are reversible! For each $i \in \{1, \dots, k\}$, let R'_i be the elementary row operation that reverses (undoes) the elementary row operation R_i . If we apply the sequence $R'_k, \dots, R'_1$ of elementary row operations to B , we obtain the matrix A . So, $B \sim A$.

Proposition 1.3.5

Let $\mathbb{F}$ be a field. Then all the following hold:

- Ⓐ for all $A \in \mathbb{F}^{n \times m}$, $A \sim A$;
- Ⓑ for all $A, B \in \mathbb{F}^{n \times m}$, if $A \sim B$, then $B \sim A$;
- Ⓒ for all $A, B, C \in \mathbb{F}^{n \times m}$, if $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof (continued). (c) Fix $A, B, C \in \mathbb{F}^{n \times m}$, and assume that $A \sim B$ and $B \sim C$. Since $A \sim B$, we know that B can be obtained by applying some sequence $R_1, \dots, R_k$ of elementary row operations to A . Similarly, since $B \sim C$, we know that C can be obtained by applying some sequence $R_{k+1}, \dots, R_{k+l}$ of elementary row operations to B . But now if we apply the sequence $R_1, \dots, R_k, R_{k+1}, \dots, R_{k+l}$ to A , we get C . $\square$

- A *zero row* of a matrix is a row in which all entries are zero, and a *non-zero row* is a row that has at least one non-zero entry.
- *Zero* and *non-zero* columns are defined analogously.
- The *leading entry* of a non-zero row is the leftmost non-zero entry of that row.

- A matrix is in *row echelon form* (or simply *echelon form*), abbreviated *REF*, if it satisfies the following two conditions:
 - ① all non-zero rows are above any zero rows;
 - ② each leading entry of a non-zero row (other than the top row) is in a column strictly to the right of the column containing the leading entry of the row right above.¹

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Here, $\blacksquare$'s represent non-zero numbers, and $*$'s represent arbitrary numbers.

¹So, all entries in a column below a leading entry of a row are zeros.

- If, in addition, the matrix satisfies the following two conditions, then it is in *reduced row echelon form* (or simply *reduced echelon form*), abbreviated RREF:
 - 3 the leading entry in each non-zero row is 1;
 - 4 each leading 1 is the only non-zero entry in its column.

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Here, *'s represent arbitrary numbers.
- If a matrix is in row echelon form (resp. reduced row echelon form), then we also say that the matrix is a *row echelon matrix* (resp. *reduced row echelon matrix*).

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

- A *pivot position* of a matrix in row echelon form is the position of a leading entry of a non-zero row, and a *pivot column* of a matrix in row echelon form is a column that contains a pivot position.
- In our diagram representing a matrix in row echelon form, the pivot positions are the positions of the black squares, and the pivot columns are the columns containing those black squares.
- In the special case of matrices in reduced row echelon form, the pivot positions are the positions of the leading 1's of the non-zero rows, and the pivot columns are the columns containing those leading 1's.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

Theorem 1.3.6

Every matrix (with entries in some field) is row equivalent to a **unique** matrix in reduced row echelon form.

- Proof: Lecture Notes (optional).
 - The proof is an example of a slightly more involved proof by induction.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

Theorem 1.3.6

Every matrix (with entries in some field) is row equivalent to a **unique** matrix in reduced row echelon form.

Corollary 1.3.7

If two row equivalent matrices (with entries in some field) are both in row echelon form, then they have exactly the same pivot positions and exactly the same pivot columns.

- Proof of Corollary 1.3.7: Lecture Notes (optional).
 - The proof of Corollary 1.3.7 is not very hard, if we assume that Theorem 1.3.6 is true.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

- By Theorem 1.3.6, every matrix A (with entries in some field) is row equivalent to a unique matrix in reduced row echelon form, which we call the *reduced row echelon form* of A , denoted by $\text{RREF}(A)$.

Corollary 1.3.8

Two matrices (with entries in some field) are row equivalent iff they have the same reduced row echelon form.

- Proof: Lecture Notes (optional).
 - The proof of Corollary 1.3.8 is not very hard, if we assume that Theorem 1.3.6 is true.

$$\begin{bmatrix}
 0 & \blacksquare & * & * & * & * & * & * & * & * \\
 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\
 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \quad
 \begin{bmatrix}
 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\
 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\
 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}$$

REF

RREF

- By Theorem 1.3.6, every matrix A (with entries in some field) is row equivalent to a unique matrix in reduced row echelon form, which we call the *reduced row echelon form* of A , denoted by $\text{RREF}(A)$.
- A *row echelon form* of a matrix A is any matrix that is in row echelon form and is row equivalent to A .
- A matrix may have more than one row echelon form (i.e. it may be row equivalent to more than one matrix in row echelon form), but by Corollary 1.3.7, all row echelon matrices of a given matrix have the same “shape,” i.e. their “black squares” are in the same place.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

- The *pivot positions* and the *pivot columns* of an arbitrary matrix A (with entries from some field) are the pivot positions and the pivot columns, respectively, of any matrix in row echelon form that is row equivalent to A .
 - By Corollary 1.3.7, this is well-defined.
- In particular, if we have computed the reduced row echelon form of a matrix A , then we can immediately identify the pivot positions and the pivot columns of A .

- We now describe an algorithm, called the *row reduction* algorithm, that transforms any matrix (with entries in some field) into a row equivalent matrix that is in reduced row echelon form.
 - This algorithm proves the existence part of Theorem 1.3.6, but not the uniqueness part.
- The algorithm has two parts: the “forward phase” and the “backward phase.”
- The forward phase transforms the matrix into one in row echelon form.
- The backward phase transforms a matrix in row echelon form into one in **reduced** row echelon form.
- In the description of the algorithm, we will use the word “pivot” to mean the actual number that is in the pivot position in question (or that we intend to move into the pivot position).
- Let's describe the algorithm!

The row reduction algorithm:

- **Forward phase:**

- ① Begin with the leftmost non-zero column. This is a pivot column. The pivot position is at the top of the column.
- ② Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
- ③ Use elementary row operations of the form " $R_j \rightarrow R_j + \alpha R_i$ " (where row i contains the pivot position in question, row j is below row i , and α is a suitable scalar) to create zeros in all positions below the pivot position.
- ④ Cover (or ignore) the row containing the pivot position, as well as all the rows (if any) above it. Apply steps 1-4 to the submatrix that remains. Repeat the process until there are no more non-zero rows to modify.

- **Backward phase:**

- ⑤ Beginning with the rightmost pivot column and working upward and to the left, create zeros above each pivot position. If a pivot is not 1, make it 1 by a scaling operation (" $R_i \rightarrow \alpha R_i$," for a suitable scalar $\alpha \neq 0$).

Example 1.3.9

Apply the row reduction algorithm to the matrix A below (with entries understood to be in $\mathbb{R}$) in order to compute its reduced row echelon form.

$$A := \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix}$$

Solution. We first implement the forward phase of the algorithm in order to transform the matrix into one in row echelon form, as follows (next slide).

Solution (continued). Forward phase:

$$A = \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 0 & -3 & -6 & 3 & 4 & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \quad \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & -3 & -6 & 3 & 4 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{3}{2}R_2 \quad \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The forward phase of the row reduction algorithm is now complete: our matrix is in row echelon form.

Solution (continued). Backward phase:

$$A \quad \sim \quad \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{array}{l} \text{by the} \\ \text{forward} \\ \text{phase} \end{array}$$

$$\begin{array}{l} R_1 \rightarrow R_1 + 6R_3 \\ R_2 \rightarrow R_2 - 2R_3 \\ \sim \end{array} \begin{bmatrix} 2 & 3 & 0 & 11 & 0 & 17 \\ 0 & -2 & -4 & 2 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow -\frac{1}{2}R_2 \\ \sim \end{array} \begin{bmatrix} 2 & 3 & 0 & 11 & 0 & 17 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{l} R_1 \rightarrow R_1 - 3R_2 \\ \sim \end{array} \begin{bmatrix} 2 & 0 & -6 & 14 & 0 & 8 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ \sim \end{array} \begin{bmatrix} 1 & 0 & -3 & 7 & 0 & 4 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Example 1.3.9

Apply the row reduction algorithm to the matrix A below (with entries understood to be in $\mathbb{R}$) in order to compute its reduced row echelon form.

$$A := \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix}$$

Solution (continued). The backward phase of row reduction is now complete: our matrix is in reduced row echelon form. Thus,

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -3 & 7 & 0 & 4 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$



Example 1.3.10

Apply the row reduction algorithm to the matrix B below (with entries understood to be in $\mathbb{Z}_3$) in order to compute its reduced row echelon form.

$$B := \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 1 \end{bmatrix}$$

Solution. We will compute keeping the addition and multiplication tables for $\mathbb{Z}_3$ (below) in mind.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Solution (continued). Forward phase (Gaussian elimination):

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{c|cccc} + & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{array}$$

$$\begin{array}{l} R_1 \leftrightarrow R_4 \\ \sim \end{array} \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{array}{c|cccc} \cdot & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{array}$$

$$\begin{array}{l} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 + R_1 \\ \sim \end{array} \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{array}{l} R_3 \rightarrow R_3 + 2R_2 \\ R_4 \rightarrow R_4 + 2R_2 \\ \sim \end{array} \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} R_4 \rightarrow R_4 + R_3 \\ \sim \end{array} \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



the matrix is in
row echelon form

$$\begin{array}{c|cccc}
 + & 0 & 1 & 2 & \\
 \hline
 0 & 0 & 1 & 2 & \\
 1 & 1 & 2 & 0 & \\
 2 & 2 & 0 & 1 &
 \end{array}$$

$$\begin{array}{c|cccc}
 \cdot & 0 & 1 & 2 & \\
 \hline
 0 & 0 & 0 & 0 & \\
 1 & 0 & 1 & 2 & \\
 2 & 0 & 2 & 1 &
 \end{array}$$

Solution (continued). Backward phase:

$$B \quad (*) \quad \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

by the forward phase
(previous slide)

$$\begin{array}{l} R_1 \rightarrow R_1 + R_3 \\ R_2 \rightarrow R_2 + R_3 \\ \sim \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



the matrix is in
reduced row
echelon form
(RREF)

$$\Rightarrow \quad \text{RREF}(B) = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$



- There are several other examples in the Lecture Notes (section 1.3.3).

- How do we solve linear systems?
- First, we form the augmented matrix of our linear system, and using row reduction, we find the reduced row echelon form of that matrix.
- Then, we “translate” this matrix (in reduced row echelon form) into the linear system that it encodes.
- The linear system that we obtain is equivalent to the one that we started with, that is, the two systems have exactly the same solution set.
- We now read off the solution set as follows.

- 1 If the rightmost column of the augmented matrix (the one to the right of the vertical dotted line) is a pivot column, then the system is inconsistent, i.e. it has no solutions.
- For example, suppose that by row reduction, we obtained the following matrix (say, with coefficients in $\mathbb{R}$).

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This matrix encodes the following linear system:

$$\begin{array}{rclcl} x_1 & - & x_3 & = & 0 \\ & x_2 & + & 5x_3 & = & 0 \\ & & & 0 & = & 1 \\ & & & 0 & = & 0 \end{array}$$

Because of the equation “ $0 = 1$,” the system is inconsistent (i.e. it has no solutions).

- ② If the rightmost column of the augmented matrix (the one to the right of the vertical dotted line) is **not** a pivot column, but all the other columns **are** pivot columns, then the system has a unique solution.
- Example: next slide!

- For example, suppose that by row reduction, we obtained the following matrix (say, with coefficients in $\mathbb{R}$).

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This matrix encodes the following linear system:

$$\begin{array}{rcl} x_1 & = & -5 \\ x_2 & = & 0 \\ x_3 & = & 3 \\ 0 & = & 0 \end{array}$$

This system is consistent and has a unique solution, which we can immediately read off, as follows.

$$\begin{array}{rcl} x_1 & = & -5 \\ x_2 & = & 0 \\ x_3 & = & 3 \end{array}$$

- ③ If the rightmost column of the augmented matrix (the one to the right of the vertical dotted line) is **not** a pivot column, and at least one of the other columns is also **not** a pivot column, then the system has more than one solution, which we read off as follows.
- The variables that correspond to the **non-pivot** columns (we call these variables *free variables*) may take **any** value; these values (called *parameters*) are denoted by letters such as r, s, t .
 - The variables that correspond to the pivot columns are called *basic*, and we solve for them in terms of our parameters.
 - This form of solution is called the *parametric form of the solution*; we will also refer to it as the *general solution*.
- ④ Example: next slide!

- For example, suppose that by row reduction, we obtained the following matrix (say, with coefficients in $\mathbb{R}$).

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & 6 & 0 \\ 0 & 0 & 1 & -1 & 7 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

This matrix encodes the linear system below.

$$\begin{array}{cccccccl} x_1 & + & 2x_2 & & & + & 5x_4 & + & 6x_5 & = & 0 \\ & & & & x_3 & - & x_4 & + & 7x_5 & = & -3 \\ & & & & & & & & 0 & = & 0 \end{array}$$

The system is consistent and has more than one solution. The variables x_2, x_4, x_5 are free, and the remaining variables are basic. We now read off the solutions as follows:

$$x_1 = -2r - 5s - 6t$$

$$x_2 = r$$

$$x_3 = s - 7t - 3$$

$$x_4 = s$$

$$x_5 = t$$

where $r, s, t \in \mathbb{R}$.

- From the previous slide:

$$x_1 = -2r - 5s - 6t$$

$$x_2 = r$$

$$x_3 = s - 7t - 3$$

$$x_4 = s$$

$$x_5 = t$$

where $r, s, t \in \mathbb{R}$.

- **Remark:** Do not forget to specify which field your parameters come from! Here, we have “ $r, s, t \in \mathbb{R}$ ” because the coefficients of our system are in $\mathbb{R}$.

- Specifying the number of solutions of a linear system:
 - An inconsistent linear system has zero solutions.
 - A consistent system may have a unique solution (i.e. exactly one solution), or it may have more than one solution.
 - A consistent system with **no free variables** has a unique solution.
 - A consistent system that has **at least one free variable** has more than one solution, since each free variable can take an arbitrary value from the field $\mathbb{F}$ in question.
 - If our field is infinite (for example, if it is $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$), then a consistent system with at least one free variable has infinitely many solutions.
 - On the other hand, if our field $\mathbb{F}$ is finite, and our linear system is consistent with exactly k free variables, then the number of solutions of our system is precisely $|\mathbb{F}|^k$ (where $|\mathbb{F}|$ is the cardinality of $\mathbb{F}$, i.e. the number of elements in $\mathbb{F}$).
 - In particular, if $\mathbb{F} = \mathbb{Z}_p$ for some prime number p , then a consistent system with exactly k free variables has exactly p^k solutions.

Example 1.3.14

Solve the linear system below (with coefficients understood to be in $\mathbb{Z}_3$), and specify how many solutions it has.

$$\begin{array}{ccccccccc} & & x_2 & + & x_3 & & & & = & 2 \\ 2x_1 & + & x_2 & & & & + & x_4 & = & 1 \\ 2x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ x_1 & & & + & 2x_3 & + & 2x_4 & = & 1 \end{array}$$

Solution. The augmented matrix of this linear system is the matrix B below.

$$B = \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 1 \end{array} \right]$$

This is precisely the matrix from Example 1.3.10. The reduced row echelon form of this matrix (computed in Example 1.3.10) is

$$\text{RREF}(B) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Solution (continued). Reminder:

$$\text{RREF}(B) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

We see from $\text{RREF}(B)$ that the rightmost column (the one to the right of the vertical dotted line) of the augmented matrix B of our linear system is **not** a pivot column; therefore, our linear system is consistent. We further see from $\text{RREF}(B)$ that the pivot columns of B are its first, second, and third column; so, the basic variables of our linear system are x_1, x_2, x_3 , whereas the remaining variable (namely, x_4) is free.

Now, $\text{RREF}(B)$ is the augmented matrix of the linear system below, which is equivalent to our original linear system.

$$\begin{array}{rclcl} x_1 & & & + & 2x_4 & = & 1 \\ & x_2 & & & & = & 2 \\ & & x_3 & & & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

Solution (continued). Reminder:

$$\begin{array}{rcccccl} x_1 & & & + & 2x_4 & = & 1 \\ & x_2 & & & & = & 2 \\ & & x_3 & & & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

Variables x_1, x_2, x_3 are basic, whereas x_4 is free.

We read off the solutions as follows:

$$\begin{array}{lcl} x_1 & = & t + 1 \\ x_2 & = & 2 \\ x_3 & = & 0 \\ x_4 & = & t \end{array} \quad \text{where } t \in \mathbb{Z}_3.$$

Our linear system is consistent and has **one** free variable. Since the field $\mathbb{Z}_3$ has **three** elements, the number of solutions is $3^1 = 3$. $\square$

- A *homogeneous linear system* is a linear system of the form

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & 0 \\ & & & & & & \vdots & & \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \dots & + & a_{n,m}x_m & = & 0 \end{array}$$

where the coefficients $a_{i,j}$ are all from some field $\mathbb{F}$ (and 0 is also understood to be from that same field $\mathbb{F}$).

- Such a system is always consistent: $x_1 = x_2 = \dots = x_m = 0$ is a solution, called the *trivial solution*.
- A *non-trivial solution* of a homogeneous linear system is a solution that is not trivial.
- Some homogeneous linear systems have only the trivial solution, whereas others also have non-trivial solutions.
 - This depends on whether there are any free variables.
- When working with homogeneous linear systems, we typically row reduce only the **coefficient matrix**, and not the augmented matrix.

Example 1.3.17

Solve the homogeneous linear system below, with coefficients understood to be in $\mathbb{R}$.

$$\begin{array}{cccccccl} 2x_1 & - & 4x_2 & & & + & 6x_4 & = & 0 \\ 2x_1 & - & 4x_2 & + & 2x_3 & - & 2x_4 & = & 0 \end{array}$$

How many solutions does this homogeneous linear system have?
Does it have any non-trivial solutions?

Solution. The **coefficient** matrix of our homogeneous linear system is

$$A := \begin{bmatrix} 2 & -4 & 0 & 6 \\ 2 & -4 & 2 & -2 \end{bmatrix}$$

We row reduce this matrix as follows (next slide):

Solution (continued).

$$A = \begin{bmatrix} 2 & -4 & 0 & 6 \\ 2 & -4 & 2 & -2 \end{bmatrix}$$

$$\underset{\sim}{R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 2 & -4 & 0 & 6 \\ 0 & 0 & 2 & -8 \end{bmatrix}$$

$$\begin{matrix} R_1 \rightarrow \frac{1}{2}R_1 \\ \underset{\sim}{R_2 \rightarrow \frac{1}{2}R_2} \end{matrix} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}.$$

The last matrix from the calculation above is in reduced row echelon form, and so

$$\text{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}.$$

Solution (continued). Reminder:

- $A = \begin{bmatrix} 2 & -4 & 0 & 6 \\ 2 & -4 & 2 & -2 \end{bmatrix},$
- $\text{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}.$

Remark:

- We must keep in mind that A is the **coefficient** matrix of our linear system.
- The **augmented** matrix of our linear system would be $\left[A \mid \mathbf{0} \right].$
- Since zero columns remain unchanged when we perform elementary row operations, the matrix $\text{RREF}\left(\left[A \mid \mathbf{0} \right]\right)$ is obtained by adding a zero column to the right of $\text{RREF}(A).$
- However, we do not normally write all this! We simply keep track of it mentally.

Solution (continued). Reminder: $\text{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$.

We now continue our computation. We see from the matrix $\text{RREF}(A)$ that the pivot columns of the coefficient matrix A are its first and third column. So, x_1, x_3 are the basic variables, and x_2, x_4 are the free variables. Further, we see from $\text{RREF}(A)$ that our original linear system is equivalent to the linear system below.

$$\begin{array}{rclclcl} x_1 & - & 2x_2 & & + & 3x_4 & = & 0 \\ & & & x_3 & - & 4x_4 & = & 0 \end{array}$$

We now read off the solutions as follows:

$$\begin{array}{rcl} x_1 & = & 2s - 3t \\ x_2 & = & s \\ x_3 & = & 4t \\ x_4 & = & t \end{array} \quad \text{where } s, t \in \mathbb{R}.$$

Solution (continued). Reminder: Our general solution was

$$x_1 = 2s - 3t$$

$$x_2 = s$$

$$x_3 = 4t$$

$$x_4 = t \quad \text{where } s, t \in \mathbb{R}.$$

Since our system has free variables (in fact, two of them), and since we are working over the infinite field $\mathbb{R}$, we see that our system has infinitely many solutions. In particular, our system has a non-trivial solution (in fact, it has infinitely many of them). $\square$

- Subsections for self-study:
 - Subsection 1.3.5: Solving systems of linear equations via back substitution
 - This is a slightly different method for solving linear systems.
 - It is not so convenient for solving linear systems by hand, but it is a method that computers use to solve linear systems, and you might be asked to program it.
 - Subsection 1.3.6: A few more remarks about the (reduced) row echelon form
 - Easy! Everything follows from the appropriate definitions and from the description of the row reduction algorithm.
 - Read it to check your understanding.
 - Subsection 1.3.7: Proof of Theorem 1.3.6 and Corollaries 1.3.7 and 1.3.8
 - Optional (for the ambitious).