Linear Algebra 1: Tutorial 12

Irena Penev

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Theorem 3.1.7 of the Lecture Notes. *Let V be a vector space over a field $\mathbb{F}$, and let $U \subseteq V$. Then U is a subspace of V if and only if the following three conditions are satisfied:*

(i) $\mathbf{0} \in U$;[1]

(ii) *U is closed under vector addition, that is, for all $\mathbf{u}, \mathbf{v} \in U$, we have that $\mathbf{u} + \mathbf{v} \in U$;*

(iii) *U is closed under scalar multiplication, that is, for all $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{u} \in U$.*

Problem 2 from HW#7. *Let V be a vector space over a field $\mathbb{F}$, and let U_1 and U_2 be subspaces of V.*

(a) *Prove that $U_1 \cap U_2$ is a subspace of V.*

(b) *Prove that $U_1 + U_2 := \{\mathbf{u}_1 + \mathbf{u}_2 \mid \mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2\}$ is a subspace of V.*

(c) *Prove that $U_1 \cup U_2$ is a subspace of V if and only if one of U_1, U_2 is included in the other.*[2]

Hint (for all three parts): *Theorem 3.1.7 of the Lecture Notes.*

Exercise 1.

(a) *Either construct a matrix $A \in \mathbb{R}^{4\times 7}$ such that Col$(A) \cong$ Nul(A) (and prove that your matrix A really does have this property), or prove that no such matrix A exists.*

(b) *Either construct a matrix $B \in \mathbb{R}^{3\times 4}$ such that Col$(B) \cong$ Nul(B) (and prove that your matrix B really does have this property), or prove that no such matrix B exists.*

[1] Here, $\mathbf{0}$ is the zero vector in the vector space V.

[2] "One of U_1, U_2 is included in the other" means "$U_1 \subseteq U_2$ or $U_2 \subseteq U_1$."

(c) *Either construct a matrix* $C \in \mathbb{R}^{3\times 8}$ *such that* $Col(C) \cong Nul(C)$ *(and prove that your matrix* C *really does have this property), or prove that no such matrix* C *exists.*

Exercise 2. *For each pair of real vector spaces* U *and* V *below, either construct a one-to-one linear function* $f : U \to V$ *or explain why no such* f *exists. If you succeed in constructing such an* f*, explain whether your* f *is also an isomorphism.*

(a) $U = \mathbb{P}^2_{\mathbb{R}}$, $V = \mathbb{R}^2$;

(b) $U = \mathbb{R}^2$, $V = \mathbb{P}^2_{\mathbb{R}}$;

(c) $U = \mathbb{P}^3_{\mathbb{R}}$, $V = \mathbb{R}^{2\times 2}$;

(d) $U = \mathbb{P}_{\mathbb{R}}$, $V = \mathbb{R}^{3\times 7}$.

Exercise 3. *Consider the following sets of matrices in* $\mathbb{Z}_3^{3\times 2}$*:*

(a) $\mathcal{A} = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 1 \end{bmatrix} \right\}$;

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right\}$.

For each of these two sets, determine whether

- *the set is linearly independent in* $\mathbb{Z}_3^{3\times 2}$;
- *the set is a spanning set of* $\mathbb{Z}_3^{3\times 2}$;
- *the set is a basis of* $\mathbb{Z}_3^{3\times 2}$.

Exercise 4. *Consider the following matrices in* $\mathbb{Z}_3^{2\times 3}$*:*

- $M_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$;
- $M_2 = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$;
- $M_3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$;
- $M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$.

Find a basis $\mathcal{B}_U$ *of* $U := Span(M_1, M_2, M_3, M_4)$*, and extend it to a basis* $\mathcal{B}$ *of* $\mathbb{Z}_3^{2\times 3}$*. Then, for each* $i \in \{1, 2, 3, 4\}$*, compute the coordinate vector* $\left[\ M_i\ \right]_{\mathcal{B}}$.