Linear Algebra 1

Tutorial #11

Computing bases of the images and preimages of subspaces under linear functions

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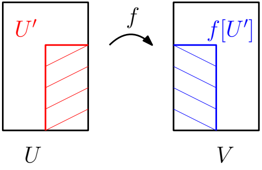
- Reminder:

Theorem 4.2.3

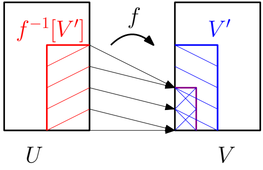
Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

a. for all subspaces U' of U, we have that $f[U']$ is a subspace of V;
b. $\text{Im}(f)$ is a subspace of V;
c. for all subspaces V' of V, we have that $f^{-1}[V']$ is a subspace of U;
d. $\text{Ker}(f)$ is a subspace of U.

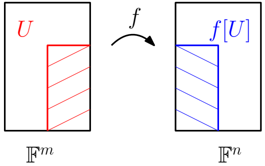
(a)



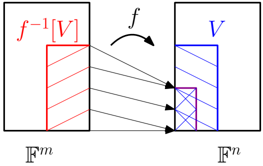
(c)



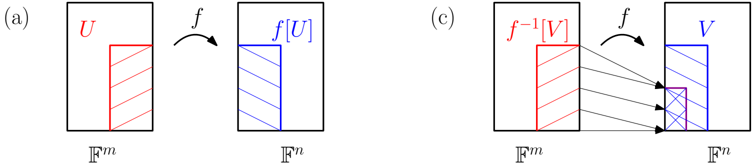
(a)



(c)



- In what follows, we will consider linear functions of the form $f : \mathbb{F}^m \to \mathbb{F}^n$ (where $\mathbb{F}$ is a field), and will give a recipe for computing:
  - a basis of the imagine of a subspace of the domain $\mathbb{F}^m$ under f (that is, a basis of $f[U]$, where U is a subspace of $\mathbb{F}^m$);
  - a basis of the preimage of a subspace of the codomain $\mathbb{F}^n$ under f (that is, a basis of $f^{-1}[V]$, where V is a subspace of $\mathbb{F}^n$).



- In what follows, we will consider linear functions of the form $f : \mathbb{F}^m \to \mathbb{F}^n$ (where $\mathbb{F}$ is a field), and will give a recipe for computing:
    - a basis of the imagine of a subspace of the domain $\mathbb{F}^m$ under f (that is, a basis of $f[U]$, where U is a subspace of $\mathbb{F}^m$);
    - a basis of the preimage of a subspace of the codomain $\mathbb{F}^n$ under f (that is, a basis of $f^{-1}[V]$, where V is a subspace of $\mathbb{F}^n$).
- We will rely on Theorem 4.2.11(b) (next slide).
    - The proof of Theorem 4.2.11 will be given during our next lecture.
    - Only part (c) of the theorem requires a bit of work to prove, but we won't use part (c) in this presentation.

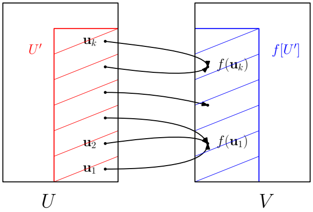
Theorem 4.2.11

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, and set $U' := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then all the following hold:

(a) U' is a subspace of U, and $f[U']$ is a subspace of V;

(b) $f[U'] = f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big] = \text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big)$, i.e. vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $f[U'] = f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big]$;

(c) $\dim\big(f[U']\big) \leq \dim(U') \leq k$.



- Informally, part (b) states: "The image of a span of vectors equals the span of the images of those vectors."

Proposition 4.2.15

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of f, let $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}^m$ ($k \geq 1$), and set $U := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then

$$f[U] \;=\; \text{Col}\Big(A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big),$$

and moreover, the pivot columns of the matrix $A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}$ form a basis of $f[U]$.

- First an example, then a proof.

Example 4.2.16

Let $f : \mathbb{Z}_2^5 \to \mathbb{Z}_2^4$ be the linear function whose standard matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix},$$

and consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{Z}_2^5$. Set $U := \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$. Find a basis for $f[U]$.

Solution.

Solution. Our goal is to find the pivot columns of the matrix $A\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix}$, since by Proposition 4.2.15, those columns form a basis of $f[U]$. First, by multiplying matrices, we obtain

$$\begin{aligned} A\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Solution (continued). By row reducing, we obtain

$$\text{RREF}\Big(A\left[\begin{array}{cccc}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4\end{array}\right]\Big) = \left[\begin{array}{cccc}1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

Solution (continued). By row reducing, we obtain

$$\text{RREF}\Big(A\left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]\Big) = \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

As we can see, the pivot columns of $A\left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]$ are its first and fourth column.

Solution (continued). By row reducing, we obtain

$$\mathrm{RREF}\Big(A\left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]\Big) = \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

As we can see, the pivot columns of $A\left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]$ are its first and fourth column. Therefore,

$$\left\{\left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \end{array}\right]\right\}$$

is a basis of $f[U]$. $\square$

Proposition 4.2.15

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n\times m}$ be the standard matrix of f, let $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}^m$ ($k \geq 1$), and set $U := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then

$$f[U] \;\; = \;\; \text{Col}\Big(A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big),$$

and moreover, the pivot columns of the matrix $A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}$ form a basis of $f[U]$.

Proof.

Proposition 4.2.15

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of f, let $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}^m$ ($k \geq 1$), and set $U := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then

$$f[U] \;=\; \text{Col}\Big(A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big),$$

and moreover, the pivot columns of the matrix $A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}$ form a basis of $f[U]$.

Proof. First, we compute (next slide):

Proof (continued).

$$\begin{aligned} f[U] &= f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big] \\ &\overset{(*)}{=} \text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big) \\ &\overset{(**)}{=} \text{Col}\Big(\begin{bmatrix} f(\mathbf{u}_1) & \dots & f(\mathbf{u}_k) \end{bmatrix}\Big) \\ &\overset{(***)}{=} \text{Col}\Big(\begin{bmatrix} A\mathbf{u}_1 & \dots & A\mathbf{u}_k \end{bmatrix}\Big) \\ &\overset{(****)}{=} \text{Col}\Big(A\begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big), \end{aligned}$$

where (*) follows from Theorem 4.2.11(b), (**) follows from the definition of the column space, and (***) follows from the fact that A is the standard matrix of f, and (****) follows from the definition of matrix multiplication.

Proof (continued).

$$
\begin{aligned}
f[U] &= f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big] \\
&\overset{(*)}{=} \text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big) \\
&\overset{(**)}{=} \text{Col}\Big(\begin{bmatrix} f(\mathbf{u}_1) & \dots & f(\mathbf{u}_k) \end{bmatrix}\Big) \\
&\overset{(***)}{=} \text{Col}\Big(\begin{bmatrix} A\mathbf{u}_1 & \dots & A\mathbf{u}_k \end{bmatrix}\Big) \\
&\overset{(****)}{=} \text{Col}\Big(A\begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big),
\end{aligned}
$$

where (*) follows from Theorem 4.2.11(b), (**) follows from the definition of the column space, and (***) follows from the fact that A is the standard matrix of f, and (****) follows from the definition of matrix multiplication. By Theorem 3.3.4, the pivot columns of a matrix form a basis of the column space of that matrix, and the result follows. □

Proposition 4.2.18

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of f, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}^n$ ($k \geq 1$), and set $V := \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ Then

$$\begin{aligned} f^{-1}[V] &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] = \mathbf{0} \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] \in \mathrm{Nul}\Big(\left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \Big) \right\}. \end{aligned}$$

- First an example, then a proof.

Example 4.2.19

Consider the linear function $f : \mathbb{R}^4 \to \mathbb{R}^5$ whose standard matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -4 & 0 \\ -2 & -3 & -6 & 1 \\ 4 & 0 & 0 & 0 \\ 2 & -1 & -2 & 0 \end{bmatrix},$$

and consider the following vectors in $\mathbb{R}^5$:

- $\mathbf{v}_1 = \begin{bmatrix} -1 & 6 & 9 & -4 & 1 \end{bmatrix}^T$;
- $\mathbf{v}_2 = \begin{bmatrix} 2 & 2 & -2 & 8 & 5 \end{bmatrix}^T$;
- $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}^T$;
- $\mathbf{v}_4 = \begin{bmatrix} 0 & -2 & -3 & -1 & -1 \end{bmatrix}^T$;
- $\mathbf{v}_5 = \begin{bmatrix} 0 & -1 & -2 & 1 & 0 \end{bmatrix}^T$;
- $\mathbf{v}_6 = \begin{bmatrix} -3 & -1 & 2 & -11 & -6 \end{bmatrix}^T$.

Set $V := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_6)$. Find a basis of $f^{-1}[V]$.

Solution. We apply Proposition 4.2.18.

Solution. We apply Proposition 4.2.18. We first form the matrix

$$C \;:=\; \left[\begin{array}{c|cccccc} A & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccccc} 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -3 \\ 0 & -2 & -4 & 0 & 6 & 2 & 0 & -2 & -1 & -1 \\ -2 & -3 & -6 & 1 & 9 & -2 & 0 & -3 & -2 & 2 \\ 4 & 0 & 0 & 0 & -4 & 8 & -1 & -1 & 1 & -11 \\ 2 & -1 & -2 & 0 & 1 & 5 & 0 & -1 & 0 & -6 \end{array}\right],$$

and we find the general solution of the matrix-vector equation

$$\underbrace{\left[\begin{array}{c|cccccc} A & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right]}_{=C} \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] \;=\; \mathbf{0},$$

where the vector $\mathbf{x}$ has four entries (because A has four columns) and the vector $\mathbf{y}$ has six entries (because we have six vectors $\mathbf{v}_1, \ldots, \mathbf{v}_6$).

Solution (continued). By row reducing, we obtain

$$\text{RREF}(C) = \left[\begin{array}{cccc:cccccc} 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 & -3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Solution (continued). By row reducing, we obtain

$$\mathrm{RREF}(C) = \left[\begin{array}{cccc:cccccc} 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 & -3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

So, the general solution of our matrix-vector equation is

$$\left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] = \left[\begin{array}{c} q - 2r + 3t \\ -2p + 3q + r - s \\ p \\ 2q + r + 2t \\ \hdashline q \\ r \\ -s \\ s \\ -t \\ t \end{array}\right], \qquad \text{where } p, q, r, s, t \in \mathbb{R}.$$

Solution (continued). By row reducing, we obtain

$$\mathrm{RREF}(C) = \left[\begin{array}{cccc:cccccc} 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 & -3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

So, the general solution of our matrix-vector equation is

$$\left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] = \left[\begin{array}{c} q - 2r + 3t \\ -2p + 3q + r - s \\ p \\ 2q + r + 2t \\ \hdashline q \\ r \\ -s \\ s \\ -t \\ t \end{array}\right], \qquad \text{where } p, q, r, s, t \in \mathbb{R}.$$

But as per Proposition 4.2.18, we only need **x**!

Solution. So, we simply ignore the part below the horizontal dotted line, and we obtain:

$$\mathbf{x} \;=\; \begin{bmatrix} q-2r+3t \\ -2p+3q+r-s \\ p \\ 2q+r+2t \end{bmatrix}, \qquad \text{where } p,q,r,s,t \in \mathbb{R}.$$

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$$\mathbf{x} = \begin{bmatrix} q - 2r + 3t \\ -2p + 3q + r - s \\ p \\ 2q + r + 2t \end{bmatrix}, \qquad \text{where } p, q, r, s, t \in \mathbb{R}.$$

By separating parameters, we obtain

$$\mathbf{x} = p \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix},$$

where $p, q, r, s, t \in \mathbb{R}$.

Solution. So, we simply ignore the part below the horizontal dotted line, and we obtain:

$$\mathbf{x} \;=\; \begin{bmatrix} q - 2r + 3t \\ -2p + 3q + r - s \\ p \\ 2q + r + 2t \end{bmatrix}, \qquad \text{where } p, q, r, s, t \in \mathbb{R}.$$

By separating parameters, we obtain

$$\mathbf{x} \;=\; p\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + q\begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix} + r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix},$$

where $p, q, r, s, t \in \mathbb{R}$.

In view of Proposition 4.2.18, we now have that (next slide):

Solution.

$$
\begin{aligned}
f^{-1}[V] &= \Bigg\{ p \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix} \mid \\
&\qquad \mid p, q, r, s, t \in \mathbb{R} \Bigg\} \\
&= \operatorname{Span}\left(\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right) \\
&= \operatorname{Col}\Bigg(\underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B} \Bigg).
\end{aligned}
$$

Solution (continued). Reminder:

$$f^{-1}[V] \quad = \quad \mathrm{Col}\Bigg(\underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B} \Bigg).$$

Solution (continued). Reminder:

$$f^{-1}[V] \;=\; \text{Col}\Big(\underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B}\Big).$$

We note that the five vectors that we obtained in the second-to-last line above are not necessarily linearly independent, and so to find an actual basis of $f^{-1}[V]$, we row reduce the matrix B and use Theorem 3.3.4. Indeed, Theorem 3.3.4 guarantees that the pivot columns of B form a basis of $\text{Col}(B) = f^{-1}[V]$.

- In fact, we can immediately see that they are not linearly independent: no five vectors in $\mathbb{R}^4$ are linearly independent (by Theorem 3.2.17(a)).
- More generally, though, the reason our computation does not necessarily yield linearly independent vectors is because we "cut off" the entries below the vertical dotted line.

Solution (continued). Reminder:

$$f^{-1}[V] \quad = \quad \mathrm{Col}\Bigg(\underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B}\Bigg).$$

Solution (continued). Reminder:

$$f^{-1}[V] \quad = \quad \mathrm{Col}\Big(\underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B}\Big).$$

By row reducing, we obtain

$$\mathrm{RREF}(B) \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 7/5 \\ 0 & 0 & 1 & 0 & -4/5 \\ 0 & 0 & 0 & 1 & 17/5 \end{bmatrix}.$$

Thus, the pivot columns of B are its leftmost four columns, and those four columns form a basis of $f^{-1}[V]$.

Solution (continued). So, our final answer is that

$$\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis of $f^{-1}[V]$. $\square$

Proposition 4.2.18

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of f, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}^n$ $(k \geq 1)$, and set $V := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ Then

$$\begin{aligned} f^{-1}[V] &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c:ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array} \right] \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array} \right] = \mathbf{0} \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array} \right] \in \text{Nul}\Big(\left[\begin{array}{c:ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array} \right] \Big) \right\}. \end{aligned}$$

Proof.

Proposition 4.2.18

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n\times m}$ be the standard matrix of f, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}^n$ $(k \geq 1)$, and set $V := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ Then

$$\begin{aligned} f^{-1}[V] &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c:cccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] = \mathbf{0} \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] \in \text{Nul}\Big(\left[\begin{array}{c:cccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \Big) \right\}. \end{aligned}$$

Proof. Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$.

Proposition 4.2.18

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n\times m}$ be the standard matrix of f, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}^n$ $(k \geq 1)$, and set $V := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ Then

$$
\begin{aligned}
f^{-1}[V] &= \left\{\mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] = \mathbf{0}\right\} \\
&= \left\{\mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] \in \text{Nul}\Big(\left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right]\Big)\right\}.
\end{aligned}
$$

Proof. Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array}\right]$. Then for all vectors $\mathbf{x} = \left[\begin{array}{ccc} x_1 & \dots & x_m \end{array}\right]^T$ in $\mathbb{F}^m$, we have the following sequence of equivalent statements (next slide):

Proof (continued).

$$
\begin{aligned}
& \mathbf{x} \in f^{-1}[V] \\
\iff \quad & f(\mathbf{x}) \in \underbrace{\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)}_{=V} \\
\overset{(*)}{\iff} \quad & A\mathbf{x} \in \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \\
\iff \quad & \underbrace{x_1\mathbf{a}_1 + \dots + x_m\mathbf{a}_m}_{=A\mathbf{x}} \in \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \\
\overset{(**)}{\iff} \quad & \exists \alpha_1, \dots, \alpha_k \in \mathbb{F} \text{ s.t. } x_1\mathbf{a}_1 + \dots + x_m\mathbf{a}_m = \alpha_1\mathbf{v}_1 + \dots + \alpha_k\mathbf{v}_k \\
\iff \quad & \exists \alpha_1, \dots, \alpha_k \in \mathbb{F} \text{ s.t. } x_1\mathbf{a}_1 + \dots + x_m\mathbf{a}_m - \alpha_1\mathbf{v}_1 - \dots - \alpha_k\mathbf{v}_k = \mathbf{0},
\end{aligned}
$$

where (*) follows from the fact that A is the standard matrix of f, and (**) follows from the definition of span.

Proof (continued).

$$\mathbf{x} \in f^{-1}[V]$$

$$\overset{\text{previous slide}}{\iff} \quad \exists \alpha_1, \ldots, \alpha_k \in \mathbb{F} \text{ s.t. } x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m - \alpha_1\mathbf{v}_1 - \cdots - \alpha_k\mathbf{v}_k = \mathbf{0}$$

$$\overset{(***)}{\iff} \quad \exists y_1, \ldots, y_k \in \mathbb{F} \text{ s.t. } x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m + y_1\mathbf{v}_1 + \cdots + y_k\mathbf{v}_k = \mathbf{0}$$

$$\iff \quad \exists y_1, \ldots, y_k \in \mathbb{F} \text{ s.t. } \left[\begin{array}{ccc|ccc} \mathbf{a}_1 & \ldots & \mathbf{a}_m & \mathbf{v}_1 & \ldots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} x_1 \\ \vdots \\ x_m \\ \hline y_1 \\ \vdots \\ y_k \end{array}\right] = \mathbf{0}$$

$$\iff \quad \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \ldots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] = \mathbf{0},$$

where (***) follows via substitution $y_i := -\alpha_i \ \forall i \in \{1, \ldots, k\}$.

Proof (continued).

$$\mathbf{x} \in f^{-1}[V]$$

$$\overset{\text{previous slide}}{\iff} \quad \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] = \mathbf{0}$$

$$\overset{(****)}{\iff} \quad \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] \in \text{Nul}\Big(\left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right]\Big),$$

where (****) follows from the definition of the null space. The result is now immediate. □