Linear Algebra 1: Tutorial 11

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Exercise 4 from Tutorial 10. *Consider the following vectors, with entries understood to be in* $\mathbb{Z}_3$*:*

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Compute a basis $\mathcal{B}$ *of Span*$(\mathbf{b}_1, \dots, \mathbf{b}_5)$*, and compute a basis* $\mathcal{C}$ *of* $\mathbb{Z}_3^4$ *that extends* $\mathcal{B}$*. Then, for each* $i \in \{1, \dots, 5\}$*, compute* $\begin{bmatrix} \mathbf{b}_i \end{bmatrix}_{\mathcal{C}}$*.*

Exercise 1. *Consider the linear function* $f : \mathbb{Z}_3^3 \to \mathbb{Z}_3^3$ *given by*

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2x_2 + x_3 \\ x_2 + x_3 \\ x_1 + x_3 \end{bmatrix} \qquad \forall x_1, x_2, x_3 \in \mathbb{Z}_3.$$

Further, consider the following vectors with entries in $\mathbb{Z}_3$*:*

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Set $U :=$ *Span*$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$*.*

(a) Compute the standard matrix A *of* f*. (You may assume that* f *is indeed linear.)*

(b) Determine whether f *is an isomorphism, and if so, compute the standard matrix of* f^{-1}*.*

(c) Compute a basis of $f[U]$*.*

(d) Compute a basis of $f^{-1}[U]$*.*

Exercise 2. *For each of the following functions f, determine whether f is linear (and prove that your answer is correct). If f is linear, determine whether it is one-to-one.*

(a) $f : \mathbb{P}_{\mathbb{R}} \to \mathbb{P}_{\mathbb{R}}$ *given by* $f\big(\sum_{k=0}^{n} a_k x^k\big) = \sum_{k=0}^{n} a_k x^{2k}$ *for all integers* $n \geq 0$ *and all* $a_0, \ldots, a_n \in \mathbb{R}$*;*

(b) $f : \mathbb{P}_{\mathbb{R}} \to \mathbb{P}_{\mathbb{R}}$ *given by* $f\big(\sum_{k=0}^{n} a_k x^k\big) = \sum_{k=0}^{n} a_k^2 x^k$ *for all integers* $n \geq 0$ *and all* $a_0, \ldots, a_n \in \mathbb{R}$.

Exercise 3. *Using the rank-nullity theorem for linear functions, explain why there is* ***no*** *isomorphism* $f : \mathbb{P}^4_{\mathbb{Z}_3} \to \mathbb{Z}_3^{2\times 2}$.

> ***Remark:*** *As we shall see (next lecture), two* ***finite-dimensional*** *vector spaces over the same field are isomorphic if and only if they have the same dimension. Since* $\mathbb{P}^4_{\mathbb{Z}_3}$ *and* $\mathbb{Z}_3^{2\times 2}$ *do not have the same dimension (what are their dimensions?), they are not isomorphic, i.e. there is no isomoprhism from one to the other. However, we have not proven this theorem yet! Prove the non-existence of the isomorphism using only the results that we have proven so far.*