# Linear Algebra 1: Tutorial 10

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**Exercise 1.** *Find bases of the column, row, and null space of each of the matrices below. Moreover, compute the rank and nullity of each matrix, and check that your answer is consistent with the rank-nullity theorem.*

*(a)* $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$*, with entries understood to be in* $\mathbb{Z}_2$*;*

*(b)* $B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix}$*, with entries understood to be in* $\mathbb{Z}_3$*;*

*(c)* $C = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -3 \\ 0 & 7 & 0 \end{bmatrix}$*, with entries understood to be in* $\mathbb{R}$*.*

**Exercise 2.** *Let* $\mathbb{F}$ *be a field. Find the dimension of each of the following vector spaces (over* $\mathbb{F}$*):*

*(a)* $\mathbb{P}^n_{\mathbb{F}}$*, where* $n$ *is a non-negative integer;*

*(b)* $\{p(x) \in \mathbb{P}^n_{\mathbb{F}} \mid p(0) = 0\}$*, where* $n$ *is a non-negative integer;*

*(c)* $\mathbb{F}^{n \times m}$*, where* $n$ *and* $m$ *are positive integers.*

> ***Remark/Hint:*** *Exhibit a basis of each vector space (you do not have to give a fully formal proof of the fact that your basis is a basis, but do make sure that it* ***is*** *in fact a basis). Experiment with small values of* $n$ *and* $m$ *first, and then generalize.*

**Exercise 3.** *For each of the following vector spaces $V$ and (ordered) sets $\mathcal{B}$, explain why $\mathcal{B}$ is **not** a basis of $V$. Try to compute as little as possible.*

*(a)* $V = \mathbb{R}^{2\times 2}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}, \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \right\}$;

*(b)* $V = \mathbb{R}^{2\times 2}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$;

*(c)* $V = \mathbb{R}^{2\times 2}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$;

*(d)* $V = \mathbb{P}^3_{\mathbb{Z}_2}$, $\mathcal{B} = \left\{ x+1, x^2+x, 0, x^3 \right\}$;

*(e)* $V = \mathbb{P}^2_{\mathbb{Z}_3}$, $\mathcal{B} = \left\{ 2x+1, x+2, 1 \right\}$.

**Exercise 4.** *Consider the following vectors, with entries understood to be in $\mathbb{Z}_3$:*

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

*Compute a basis $\mathcal{B}$ of Span$(\mathbf{b}_1, \ldots, \mathbf{b}_5)$, and compute a basis $\mathcal{C}$ of $\mathbb{Z}_3^4$ that extends $\mathcal{B}$. Then, for each $i \in \{1, \ldots, 5\}$, compute $\left[\ \mathbf{b}_i\ \right]_{\mathcal{C}}$.*