Linear Algebra 1: Tutorial 9

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Problem 3 from HW#5. *Consider the following vectors, with entries understood to be in* $\mathbb{Z}_2$*:*

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Determine if there exists a linear function $f : \mathbb{Z}_2^3 \to \mathbb{Z}_2^4$ *that satisfies the property that*

$$f(\mathbf{b}_i) = \mathbf{c}_i \quad \textit{for all } i \in \{1, 2, 3, 4\}.$$

If such a linear function f *exists, determine whether it is unique, and if it is unique, find a formula for* f*.*

> ***Remark:*** *If the linear function* f *exists and is unique, then your formula for it should be of the form*
>
> $$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \qquad \textit{for all } x_1, x_2, x_3 \in \mathbb{Z}_2,$$
>
> *where the question marks are replaced with the appropriate values. If the linear function* f *does not exist, or if it exists but is not unique, then you do not need to find such a formula.*
>
> ***Hint:*** *This problem essentially boils down to solving a matrix equation of the form* $XA = B$*. Several examples very similar to this problem are given in subsection 1.10.4 of the Lecture Notes.*

Exercise 1. *Read off the solutions of the equation $AX = B$ (with matrix entries in the specified field $\mathbb{F}$) if $RREF\big(\left[\begin{array}{c|c} A & B \end{array}\right]\big)$ is as specified below.*

(a) $\mathbb{F} = \mathbb{R}$, $RREF\big(\left[\begin{array}{c|c} A & B \end{array}\right]\big) = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 5 & 0 & -5 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$;

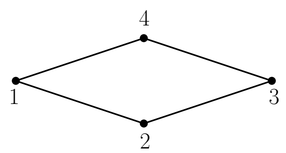
(b) $\mathbb{F} = \mathbb{Z}_3$, $RREF\big(\left[\begin{array}{c|c} A & B \end{array}\right]\big) = \left[\begin{array}{cccc|cc} 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(c) $\mathbb{F} = \mathbb{Z}_2$, $RREF\big(\left[\begin{array}{c|c} A & B \end{array}\right]\big) = \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$;

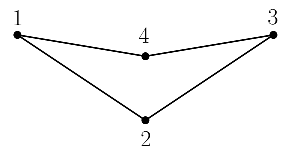
(d) $\mathbb{F} = \mathbb{Z}_2$, $RREF\big(\left[\begin{array}{c|c} A & B \end{array}\right]\big) = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$;

(e) $\mathbb{F} = \mathbb{R}$, $RREF\big(\left[\begin{array}{c|c} A & B \end{array}\right]\big) = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{array}\right]$.

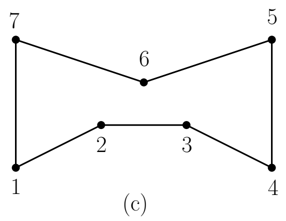
Exercise 4 from Tutorial 8. *Compute the group of symmetries for each of the polygons below.*



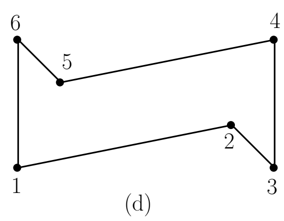
(a)



(b)



(c)



(d)

Exercise 2. *Consider the following sets $\mathcal{B}$, with entries understood to be in $\mathbb{Z}_3$.*

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\};$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\};$

(c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\};$

(d) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\};$

(e) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\};$

(f) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} \right\}.$

For each $\mathcal{B}$ do all the following.

(1) Determine whether $\mathcal{B}$ is a linearly independent set in $\mathbb{Z}_3^4$.

(2) Determine whether $\mathcal{B}$ is a spanning set of $\mathbb{Z}_3^4$.

(3) Determine whether $\mathcal{B}$ is a basis of $\mathbb{Z}_3^4$.

(4) If $\mathcal{B}$ is a basis of $\mathbb{Z}_3^4$, then compute the coordinate vector $\left[\ \mathbf{x}\ \right]_{\mathcal{B}}$, where $\mathbf{x} = \left[\ 2 \quad 1 \quad 1 \quad 1\ \right]^T$.

Exercise 3. *Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$ be a matrix, and let Nul(A) (called the* null space *of A) be the solution set of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$. Prove that Nul(A) is a subspace of $\mathbb{F}^m$.*