Linear Algebra 1: Tutorial 8

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Winter 2024

Exercise 1 from Tutorial 7. *Let $(G, \cdot)$ be a finite group,*[1] *and let H be a subgroup of G.*[2] *As usual, for $a, b \in G$, we write ab instead of $a \cdot b$. The identity element of G is denoted by e, and the inverse of an element $a \in G$ is denoted by a^{-1}. For all $a \in G$, define*

$$aH \;\; := \;\; \{ah \mid h \in H\}.$$

For any finite set X, we denote by $|X|$ the cardinality (i.e. the number of elements) of X.

(a) Prove that for all $a \in G$, we have that $a \in aH$ (and in particular, $aH \neq \emptyset$).

(b) Prove that for all $a, b \in G$, either $aH = bH$ or $aH \cap bH = \emptyset$.

(c) Prove that there exist some $a_1, \dots, a_k \in G$ such that

$$(a_1H, \dots, a_kH)$$

is a partition of G, that is, such that $G = a_1H \cup \dots \cup a_kH$ and such that $a_1H, \dots, a_kH$ are pairwise disjoint.

(d) Prove that for all $a \in G$, we have that $|aH| = |H|$, that is, aH and H have the same number of elements.

> ***Hint:*** *Create a bijection from H to aH (and prove that it really is a bijection).*

(e) Using the previous parts, prove that $|H| \mid |G|$, that is, that $|G|$ is divisible by $|H|$.

[1] This means that $|G|$ is finite, i.e. G has only finitely many elements.

[2] Technically, $(H, \cdot)$ is a subgroup of $(G, \cdot)$.

Exercise 1. *Consider the following permutation in S_5:*

$$\pi \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}.$$

(a) Compute the disjoint cycle decomposition of π.

(b) Express π as a composition of transpositions.

(c) Find all the inversions of π.

(d) Compute the sign of π in three different ways:

- *using the definition (i.e. using the disjoint cycle decomposition of π),*
- *using transpositions,*
- *using inversions.*

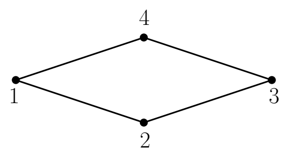
(e) Compute π^{-1} and sgn(π^{-1}). Give both the table form and the disjoint cycle decomposition of π^{-1}.

Exercise 2. *Repeat the previous exercise, only for the following permutation in S_7:*

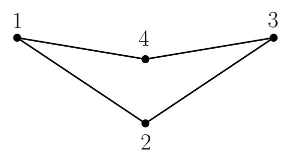
$$\pi \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 2 & 3 & 7 & 5 & 6 \end{pmatrix}.$$

Exercise 3. *Consider the permutations $\sigma = (143)(26)$ and $\pi = (27345)$ in S_7. Compute sgn(σ), sgn(π), sgn$(\sigma \circ \pi)$, and sgn$(\pi \circ \sigma)$.*[3] *Then, compute $\sigma \circ \pi$ and $\pi \circ \sigma$. Give both the disjoint cycle representations and the table forms of $\sigma \circ \pi$ and $\pi \circ \sigma$.*

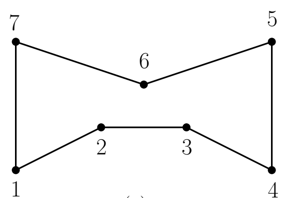
Exercise 4. *Compute the group of symmetries for each of the polygons below.*



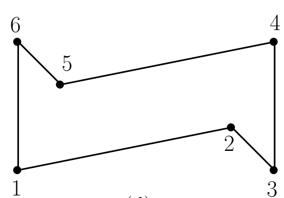
(a)



(b)



(c)



(d)

[3] You should be able to compute sgn$(\sigma \circ \pi)$ and sgn$(\pi \circ \sigma)$ without actually computing the permutations $\sigma \circ \pi$ and $\pi \circ \sigma$.