Linear Algebra 1: Tutorial 7

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Exercise 3 from Tutorial 6. *Consider the matrix below, with entries understood to be in* $\mathbb{R}$*.*

$$A \;=\; \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Either express A *as a product of elementary matrices, or prove that this is not possible.*

Exercise 4 from Tutorial 6. *Let* $\mathbb{F}$ *be a field, and let* $A \in \mathbb{F}^{n\times m}$*. Set*

$$\left[\begin{array}{c:c} U & C \end{array}\right] \;=\; RREF\Big(\left[\begin{array}{c:c} A & I_n \end{array}\right]\Big).$$

(Here, U *is an* $n \times m$ *matrix, and* C *is an* $n \times n$ *matrix.) What is the relationship between* A*,* U*, and* C*?*

Exercise 5 from Tutorial 6. *Consider the matrix*

$$A \;:=\; \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_2$*. Compute an invertible matrix* $C \in \mathbb{Z}_2^{3\times 3}$ *such that* $RREF(A) = CA$*.*

Exercise 6 from Tutorial 6. *For each of the following matrices* A *and* B*, either compute an invertible matrix* C *such that* $B = CA$*, or prove that no such matrix* C *exists.*

(a) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$*, with entries in* $\mathbb{Z}_2$*.*

(b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, *with entries in* $\mathbb{Z}_3$.

Exercise 7 from Tutorial 6. *Construct two invertible matrices $A, B \in \mathbb{R}^{2\times 2}$ such that $A + B$ is* ***not*** *invertible.*

Exercise 1. *Let $(G, \cdot)$ be a finite group,*[1] *and let H be a subgroup of G.*[2] *As usual, for $a, b \in G$, we write ab instead of $a \cdot b$. The identity element of G is denoted by e, and the inverse of an element $a \in G$ is denoted by a^{-1}. For all $a \in G$, define*

$$aH \quad := \quad \{ah \mid h \in H\}.$$

For any finite set X, we denote by $|X|$ the cardinality (i.e. the number of elements) of X.

(a) Prove that for all $a \in G$, we have that $a \in aH$ (and in particular, $aH \neq \emptyset$).

(b) Prove that for all $a, b \in G$, either $aH = bH$ or $aH \cap bH = \emptyset$.

(c) Prove that there exist some $a_1, \ldots, a_k \in G$ such that

$$(a_1H, \ldots, a_kH)$$

is a partition of G, that is, such that $G = a_1H \cup \cdots \cup a_kH$ and such that $a_1H, \ldots, a_kH$ are pairwise disjoint.

(d) Prove that for all $a \in G$, we have that $|aH| = |H|$, that is, aH and H have the same number of elements.

> ***Hint:*** *Create a bijection from H to aH (and prove that it really is a bijection).*

(e) Using the previous parts, prove that $|H| \mid |G|$, that is, that $|G|$ is divisible by $|H|$.

[1] This means that $|G|$ is finite, i.e. G has only finitely many elements.
[2] Technically, $(H, \cdot)$ is a subgroup of $(G, \cdot)$.