Linear Algebra 1: Tutorial 6

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Exercise 2 from HW#2. *Solve the following system of linear equations, with coefficients understood to be in* $\mathbb{Z}_3$. *How many solutions does the system have?*

$$\begin{array}{rcrcrcrcrcl} 2x_1 & + & x_2 & + & 2x_3 & + & 2x_4 & + & x_5 & = & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & + & x_5 & = & 2 \\ 2x_1 & + & x_2 & & & & & + & 2x_5 & = & 2 \end{array}$$

Problem 2 from HW#2. *For which (if any) values of the real parameter* k *are the real matrices*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} k & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

row equivalent? Make sure you prove that your answer is correct.

> ***Hint:*** *There is a result in the Lecture Notes that tells you when two matrices are row equivalent. Which result (proposition/lemma/theorem/corollary) is that?*

Exercise 1. *Determine which (if any) of the following matrices are invertible. If a matrix is invertible, find its inverse.*

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ *with entries understood to be in* $\mathbb{R}$.

(b) $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ *with entries understood to be in* $\mathbb{Z}_2$.

(c) $C = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ *with entries understood to be in* $\mathbb{Z}_3$.

Exercise 2. *Consider the following* ***elementary*** *matrices (with entries understood to be in* $\mathbb{R}$*).*

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For each of the matrices above, determine which elementary row operation it corresponds to, and find the inverse of the matrix. (You should be able to find the inverse at a glance, without any row reducing.)

Exercise 3. *Consider the matrix below, with entries understood to be in* $\mathbb{R}$*.*

$$A \quad = \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Either express A *as a product of elementary matrices, or prove that this is not possible.*

Exercise 4. *Let* $\mathbb{F}$ *be a field, and let* $A \in \mathbb{F}^{n \times m}$*. Set*

$$\left[\, U \,\vdots\, C \,\right] \quad = \quad RREF\Big(\left[\, A \,\vdots\, I_n \,\right] \Big).$$

(Here, U *is an* $n \times m$ *matrix, and* C *is an* $n \times n$ *matrix.) What is the relationship between* A*,* U*, and* C*?*

Exercise 5. *Consider the matrix*

$$A \quad := \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_2$*. Compute an invertible matrix* $C \in \mathbb{Z}_2^{3\times 3}$ *such that* $RREF(A) = CA$*.*

Exercise 6. *For each of the following matrices* A *and* B*, either compute an invertible matrix* C *such that* $B = CA$*, or prove that no such matrix* C *exists.*

(a) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$*, with entries in* $\mathbb{Z}_2$*.*

(b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$*, with entries in* $\mathbb{Z}_3$*.*

Exercise 7. *Construct two invertible matrices* $A, B \in \mathbb{R}^{2\times 2}$ *such that* $A + B$ *is* ***not*** *invertible.*