Linear Algebra 1: Tutorial 5

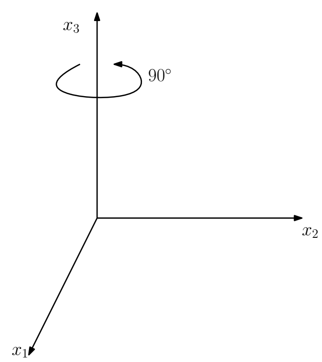
Irena Penev

Winter 2024

Problem 4 from HW#1. *Show that any collection of at least 5 cities can be connected via one-way flights*[1] *in such a way that any city is reachable from any other city with at most one layover.*

Exercise 1. *Find the standard matrices of the following linear functions (you may assume they are linear).*

(a) $f : \mathbb{R}^3 \to \mathbb{R}^3$ *that rotates each vector about the* x_3*-axis by 90° counterclockwise.*



(b) $g : \mathbb{R}^3 \to \mathbb{R}^3$ *that projects each vector onto the* x_1x_3*-plane.*

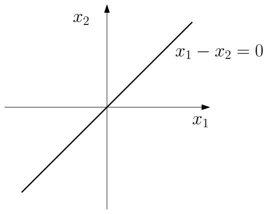
(c) $h_1 : \mathbb{R}^3 \to \mathbb{R}^3$ *that first rotates each vector about the* x_3*-axis by 90° counterclockwise, and then projects each vector onto the* x_1x_3*-plane.*

(d) $h_2 : \mathbb{R}^3 \to \mathbb{R}^3$ *that first projects each vector onto the* x_1x_3*-plane, and then rotates each vector about the* x_3*-axis by 90° counterclockwise.*

[1] So, for any two cities, A and B, at most one of the following (direct) flights is possible:

- from A to B;
- from B to A.

Exercise 2. *Find the standard matrix of the linear function* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *that projects each vector onto the line given by the equation* $x_1 - x_2 = 0$ *(you may assume the function is linear).*



Exercise 3. *Determine which (if either) of the following is a linear function, and prove that your answer is correct. If the function is linear, compute its standard matrix.*

(a) $f : \mathbb{Z}_2^2 \to \mathbb{Z}_2^4$ *given by* $f\Big(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T\Big) = \begin{bmatrix} x_1 & x_1^2 & x_2^3 & x_2^4 \end{bmatrix}^T$ *for all* $x_1, x_2 \in \mathbb{Z}_2$.

(b) $g : \mathbb{Z}_3^3 \to \mathbb{Z}_3^3$ *given by* $g\Big(\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T\Big) = \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 \end{bmatrix}^T$ *for all* $x_1, x_2, x_3 \in \mathbb{Z}_3$.

Hint: *One of these (which one?) is a trick question.*

Exercise 4. *Let*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

with entries understood to be in $\mathbb{Z}_2$*. Solve the matrix equation* $AX = B$*. How many solutions does the equation* $AX = B$ *have?*

Exercise 5. *Let*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \textit{array} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

with entries understood to be in $\mathbb{R}$*. Solve the matrix equation* $AX = B$*. How many solutions does the equation* $AX = B$ *have?*

Exercise 6. *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 2 \end{bmatrix}$$

with entries understood to be in $\mathbb{Z}_3$*. Solve the matrix equation* $XA = B$*. How many solutions does the equation* $XA = B$ *have?*