Linear Algebra 1: Tutorial 4

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Exercise 6 from Tutorial 3. *Consider the following vectors in* $\mathbb{Z}_2^4$*:*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

For each of the vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}$*, determine if it can be expressed as a linear combination of the vectors* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ *(that is, if it belongs to* $Span(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$*), and if so, express it as such a linear combination, and explain whether your answer is unique.*

Exercise 1. *Consider the real matrices* A *and* B *below. Compute the following matrices:* A^T*,* B^T*,* AB*,* BA*.*

$$A = \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 3 & -2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ -2 & 0 \\ 3 & 1 \\ 2 & 2 \end{bmatrix}.$$

Exercise 2. *Consider matrices* A, B, C *below, with entries understood to be in* $\mathbb{Z}_2$*. Compute the matrix* ABC*.*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Exercise 3. *Consider the square matrix* A *below, with entries understood to be in* $\mathbb{Z}_3$*. Compute* A^2 *and* A^3*.*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Definition. *A* symmetric matrix *is any square matrix* A *such that* $A^T = A$.

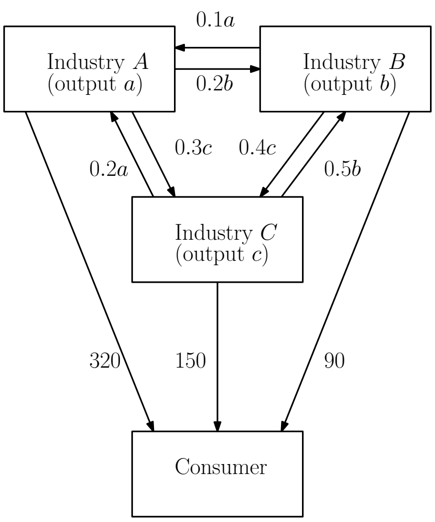
Exercise 4. *Let* $\mathbb{F}$ *be a field, and let* $A \in \mathbb{F}^{n\times m}$. *Explain why* AA^T *and* $A^T A$ *are both defined, and explain why they are both symmetric matrices.*

Exercise 5. *Construct non-zero matrices* A *and* B *(with entries in a field* $\mathbb{F}$ *of your choice) such that* $AB = O$ *(where* O *is the zero matrix of the appropriate size). Can you construct* A *and* B *so that neither of them has* ***any*** *zero entries?*

Exercise 6. *Construct a non-zero square matrix* A *(with entries in a field of your choice) such that* $A^2 = O$ *(where* O *is the zero matrix of the appropriate size).*

> ***Terminology:*** *A square matrix* A *such that* $A^m = O$ *for some positive integer* m *is called* nilpotent.

Exercise 7. *An economy has three industries,* A, B, *and* C, *which have annual output worth* a, b, *and* c, *respectively (this is measured in some units, such as hundreds of millions of dollars). The consumer requires a certain amount of output from each industry (as shown by the arrows). However, in order to produce output, each industry requires input from the other two industries, as shown by the arrows. (For instance, if industry* A *is to produce total output valued* a, *it needs to receive input valued* $0.1a$ *from industry* B, *as well as input valued* $0.2a$ *from industry* C.*) Compute the output values* a, b, c *so that the requirements of both the consumer and the three industries are satisfied.*



> ***Remark:*** *The calculation gets messy, and so feel free to use software for row reduction.*