Linear Algebra 1: Tutorial 3

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$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Exercise 1. *Consider the matrix and vector below, with entries understood to be in* $\mathbb{Z}_3$*.*

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$*. How many solutions does this equation have?*

Exercise 2. *Consider the matrix and vector below, with entries understood to be in* $\mathbb{R}$*.*

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 3 & 2 \\ -2 & 6 & -2 \\ 3 & -9 & 0 \\ 1 & -3 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$*. How many solutions does this equation have?*

Exercise 3. *Consider the matrix and vector below, with entries understood to be in* $\mathbb{Z}_2$*.*

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$*. How many solutions does this equation have?*

Exercise 4. *Either construct a matrix* A *and vectors* $\mathbf{b}, \mathbf{c}$ *that satisfy the given specifications, or explain why such* $A, \mathbf{b}, \mathbf{c}$ *do not exist. In each case, the entries of* $A, \mathbf{b}, \mathbf{c}$ *should be from the same field, and the number of rows of* A *should be the same as the number of entries in each of the vectors* $\mathbf{b}, \mathbf{c}$*.*

(a) $A\mathbf{x} = \mathbf{b}$ *has no solutions, whereas* $A\mathbf{x} = \mathbf{c}$ *has a unique solution.*

(b) $A\mathbf{x} = \mathbf{b}$ *has no solutions, whereas* $A\mathbf{x} = \mathbf{c}$ *has more than one solution.*

(c) $A\mathbf{x} = \mathbf{b}$ *has a unique solution, whereas* $A\mathbf{x} = \mathbf{c}$ *has more than one solution.*

*(**Hint:** Think about rank, pivot/non-pivot columns, and free variables.)*

Exercise 5. *Consider the following vectors in* $\mathbb{R}^3$*:*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine whether $\mathbf{b} \in Span(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$*. If so, express* $\mathbf{b}$ *as a linear combination of the vectors* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$*, and explain whether your answer is unique.*

Exercise 6. *Consider the following vectors in* $\mathbb{Z}_2^4$*:*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

For each of the vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}$*, determine if it can be expressed as a linear combination of the vectors* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ *(that is, if it belongs to* $Span(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$*), and if so, express it as such a linear combination, and explain whether your answer is unique.*