Linear Algebra 1: Tutorial 1

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Exercise 1. *Compute the following:*

(a) $1 + 1 + 1 - 1 + 1$ *in* $\mathbb{Z}_2$;

(b) $2 \cdot 2 - 1$ *in* $\mathbb{Z}_3$;

(c) $1 + 2 + 1 + 2 + 1 + 2$ *in* $\mathbb{Z}_3$;

(d) $2 \cdot 3 + 4 - 3 \cdot 1 + 2 \cdot 2$ *in* $\mathbb{Z}_5$;

(e) $1 \cdot 2 \cdot 3 \cdot 4$ *in* $\mathbb{Z}_5$.

Exercise 2. *By consulting the appropriate addition and multiplication tables, solve the following equations:*

(a) $x + 1 = 0$ *in* $\mathbb{Z}_2$;

(b) $x + 1 = 0$ *in* $\mathbb{Z}_3$;

(c) $x + 1 = 0$ *in* $\mathbb{Z}_5$;

(d) $2x + 1 = 0$ *in* $\mathbb{Z}_3$;

(e) $2x + 1 = 0$ *in* $\mathbb{Z}_5$;

(f) $2x + 2 = 1$ *in* $\mathbb{Z}_3$;

(g) $2x + 2 = 1$ *in* $\mathbb{Z}_5$;

(h) $3x + 2 = 0$ *in* $\mathbb{Z}_5$;

(i) $x^2 + 1 = 0$ *in* $\mathbb{Z}_2$;

(j) $x^2 + 1 = 0$ *in* $\mathbb{Z}_3$;

(k) $x^2 + 1 = 0$ *in* $\mathbb{Z}_5$;

(l) $2x^3 + 2 = 0$ *in* $\mathbb{Z}_5$.

Remark: Exercises 3 and 4 are due to Alexander Shen (`https://mccme.ru/shen/induction.pdf`), and are reproduced here with cosmetic modifications.

Exercise 3. *A drill sergeant facing a row of new recruits shouts "Left turn!" Due to inexperience, each recruit turns left or right, at random. After one second, each recruit who sees another recruit's face realizes that something went wrong and turns the other way. The procedure continues each second until no-one is face-to-face with anyone else. Prove that at some point, the turning ends. (Hint: Induction on the number of recruits.)*

Exercise 4 (challenge problem)**.** *A businessman has a certain amount of money (a whole number of dollars) in cash when he makes the following deal with the Devil: he can trade any one of his bills for an arbitrary (finite) number of smaller value bills from the Devil. For instance, the businessman may trade one* \$100 *bill for a thousand* \$50 *bills, plus a million* \$20 *bills. The trade can take place as often as the the businessman likes. However, the businessman has no means of making money other than by trading with the Devil (though he can spend it elsewhere), and he must spend* \$1 *on food each day.*[1] *Using (strong) induction, prove that the businessman will eventually starve.*

> ***A guide for the creative, nitpicky, and/or perplexed:*** *For the purposes of this exercise, you may assume that there exist bills worth an arbitrary number of whole dollars (for instance, there is a* \$47692 *bill), but no other kinds of money exist (for example, there is no* 10¢ $=$ \$0.10*, and there is definitely no* $\$\sqrt[7]{13}$ *or* $\$\pi$*). Moreover, when the businessman goes shopping, he can receive change, but he must always use only one of his bills to pay, since otherwise, he will annoy the cashier and get himself kicked out of the store. (For instance, he can use one* \$100 *bill to pay for his food and receive* \$99 *change, but he* ***cannot*** *use one* \$100 *bill plus one* \$101 *bill to pay and receive a* \$200 *bill as change.) Moreover, the businessman must eat in order to live (and he cannot get food without paying for it), but is immortal otherwise.*

[1] Well... Maybe these are 1950 dollars.