Linear Algebra 1

Lecture #11

Linear functions (part II)

Irena Penev

January 8, 2024

- This lecture has four parts:

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    1. The effect of a linear function on linearly independent and spanning sets

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  2. Linear functions and bases

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  3. Isomorphisms

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  1. The effect of a linear function on linearly independent and spanning sets
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  3. Isomorphisms
  4. An application of isomorphisms: transforming polynomials and matrices into vectors

1. The effect of a linear function on linearly independent and spanning sets

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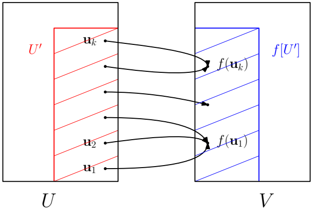
Theorem 4.2.11

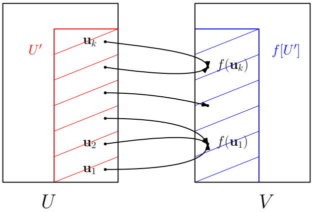
Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, and set $U' := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then all the following hold:

(a) U' is a subspace of U, and $f[U']$ is a subspace of V;

(b) $f[U'] = f\left[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\right] = \text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big)$, i.e. vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $f[U'] = f\left[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\right]$;

(c) $\dim\big(f[U']\big) \leq \dim(U') \leq k$.





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Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, and set $U' := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then all the following hold:

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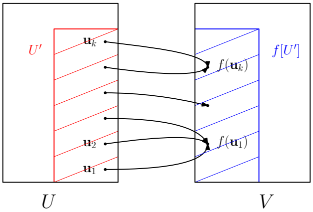
Proof of (a).



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(a) U' is a subspace of U, and $f[U']$ is a subspace of V;

Proof of (a). The fact that U' is a subspace of U follows immediately from Theorem 3.1.11, and the fact that $f[U']$ is a subspace of V follows from 4.2.3(a). This proves (a).

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(b) $f[U'] = f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big] = \text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big)$, i.e. vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $f[U'] = f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big]$;

Proof of (b).

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Proof of (b).

$$
\begin{aligned}
\mathrm{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big) &= \big\{\alpha_1 f(\mathbf{u}_1) + \dots + \alpha_k f(\mathbf{u}_k) \mid \alpha_1, \dots, \alpha_k \in \mathbb{F}\big\} \\
&\overset{(*)}{=} \Big\{f\big(\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k\big) \mid \alpha_1, \dots, \alpha_k \in \mathbb{F}\Big\} \\
&\overset{(**)}{=} \big\{f(\mathbf{u}) \mid \mathbf{u} \in \mathrm{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big\} \\
&= f\big[\mathrm{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big] \;=\; f[U'],
\end{aligned}
$$

where (*) follows from the linearity of the f (and more precisely, from Prop. 4.1.5), and (**) follows from the definition of span.

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(c) $\dim\big(f[U']\big) \leq \dim(U') \leq k$.

Proof of (c).

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- (c) $\dim\big(f[U']\big) \leq \dim(U') \leq k$.

Proof of (c). By hypothesis, $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a spanning set of U'.

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Proof of (c). By hypothesis, $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a spanning set of U'. So, by Theorem 3.2.14, some subset of that spanning set, say $\{\mathbf{u}_{i_1}, \dots, \mathbf{u}_{i_m}\}$ (with $1 \leq i_1 < \dots < i_m \leq k$) is a basis of U'.

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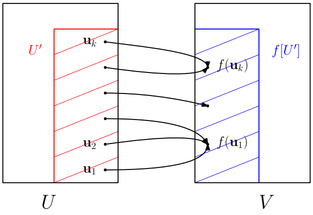
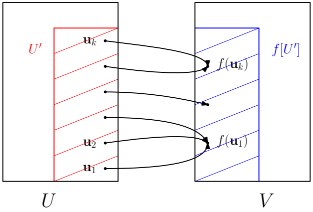
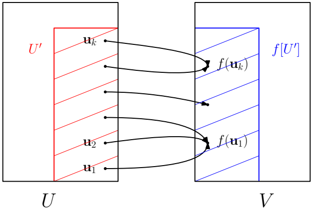




Corollary 4.2.12

Let U and V be vector spaces over a field $\mathbb{F}$, let $f : U \to V$ be a linear function, and let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a spanning set of U. Then $\text{Im}(f) = \text{Span}(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k))$ and
$\text{rank}(f) = \dim\Big(\text{Span}(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k))\Big) \leq k$.

Proof.



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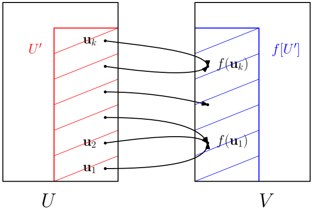
Proof. By hypothesis, $U = \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$.



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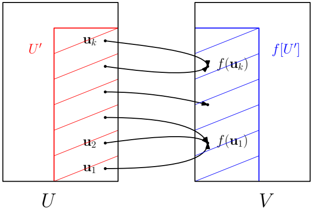
Proof. By hypothesis, $U = \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$. So, by Theorem 4.2.11(b), we have that
$\text{Im}(f) = f[U] = \text{Span}(f(\mathbf{u}_1), \ldots, f(\mathbf{u}_k))$,



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Proof. By hypothesis, $U = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. So, by Theorem 4.2.11(b), we have that $\text{Im}(f) = f[U] = \text{Span}(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k))$, and by Theorem 4.2.11(c), we have that $\text{rank}(f) = \dim(\text{Im}(f)) = \dim(f[U]) \leq k$. □

Theorem 4.2.13

Let U and V be vector spaces over a field $\mathbb{F}$, let $f : U \to V$ be a linear function, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$. Then all the following hold:

(a) if f is one-to-one and vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly independent in U, then vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ are linearly independent in V;

(b) if vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ are linearly independent in V, then vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly independent in U;

(c) if f is onto and vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ span U, then vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span V;

(d) if f is one-to-one and vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span V, then vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ span U.

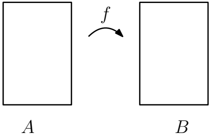
- Proof: Lecture Notes.
- Informal summary: next slide.

- Theorem 4.2.13 (schematically and informally):

$$f : U \xrightarrow{\text{linear}} V$$

(a)-(b)	$\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent	$\overset{\text{if } f \text{ is 1-1}}{\Longrightarrow}$ $\underset{\text{always}}{\Longleftarrow}$	$f(\mathbf{u}_1), \ldots, f(\mathbf{u}_k)$ are linearly independent
(c)-(d)	$\mathbf{u}_1, \ldots, \mathbf{u}_k$ span U	$\overset{\text{if } f \text{ is onto}}{\Longrightarrow}$ $\underset{\text{if } f \text{ is 1-1}}{\Longleftarrow}$	$f(\mathbf{u}_1), \ldots, f(\mathbf{u}_k)$ span V

- **Dimension considerations:**



  - As we know, for any function $f : A \to B$, where A and B are finite sets, the following hold:
    - if f is one-to-one, then $|A| \leq |B|$;
    - if f is onto, then $|A| \geq |B|$;
    - if f is a bijection, then $|A| = |B|$.

    (Actually, the above is true even if we allow A and B to be infinite, but to make sense of the statement, we would need infinite cardinals. We omit the details.)

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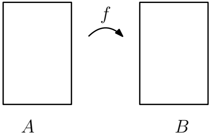

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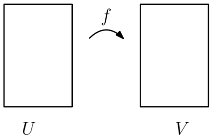
    (Actually, the above is true even if we allow A and B to be infinite, but to make sense of the statement, we would need infinite cardinals. We omit the details.)
  - In the case of **linear** functions, Theorem 4.2.14 (next slide) gives us a very similar statement, only involving dimension (rather than cardinality) of the domain and codomain.

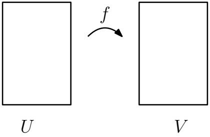


Theorem 4.2.14

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

a) if f is one-to-one, then $\dim(U) \leq \dim(V)$;
b) if f is onto, then $\dim(U) \geq \dim(V)$;
c) if f is an isomorphism, then $\dim(U) = \dim(V)$.

Proof.



Theorem 4.2.14

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

- (a) if f is one-to-one, then $\dim(U) \leq \dim(V)$;
- (b) if f is onto, then $\dim(U) \geq \dim(V)$;
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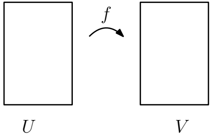
Proof. Obviously, (a) and (b) together imply (c).



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Proof. Obviously, (a) and (b) together imply (c). So, it is enough to prove (a) and (b).

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Proof (continued). (a) We prove the contrapositive: we assume that $\dim(U) > \dim(V)$ (and in particular, $\dim(V)$ is finite), and we prove that f is **not** one-to-one.

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Set $n := \dim(V)$.

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Set $n := \dim(V)$. Since $\dim(U) > \dim(V)$, we know that U has a linearly independent set of size greater than n.

- Indeed, if U is finite-dimensional, then any one of its bases is a linearly independent set of size $\dim(U) > n$, and if U is infinite-dimensional, then Proposition 3.2.18 guarantees that U has linearly independent sets of any finite size.

So, fix a linearly independent set $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U, with $k > n$.

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So, fix a linearly independent set $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U, with $k > n$. Since $\dim(V) = n$, Theorem 3.2.17(a) guarantees that the set $\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\}$ is linearly dependent. But now Theorem 4.2.13(a) guarantees that f is not one-to-one.

Theorem 4.2.14

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(b) if f is onto, then $\dim(U) \geq \dim(V)$;

Proof (continued). (b) Assume that f is onto; we must show that $\dim(U) \geq \dim(V)$.

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Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

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Proof (continued). (b) Assume that f is onto; we must show that $\dim(U) \geq \dim(V)$. We may assume that $n := \dim(U)$ is finite, for otherwise, we are done.

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Fix any basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of U. In particular, vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ span U, and so since f is onto, Theorem 4.2.13(c) guarantees that vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_n)$ span V.

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Fix any basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of U. In particular, vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ span U, and so since f is onto, Theorem 4.2.13(c) guarantees that vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_n)$ span V. But then by Theorem 3.2.14, some subset of $\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_n)\}$ is a basis of V, and it follows that $\dim(V) \leq n$. $\square$

Theorem 4.2.14

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) if f is one-to-one, then $\dim(U) \leq \dim(V)$;

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2. Linear functions and bases

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- Reminder:

Theorem 1.10.5

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be any vectors in $\mathbb{F}^n$. Then there exists a **unique** linear function $f : \mathbb{F}^m \to \mathbb{F}^n$ that satisfies $f(\mathbf{e}_1) = \mathbf{a}_1, \dots, f(\mathbf{e}_m) = \mathbf{a}_m$, where $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of $\mathbb{F}^m$. Moreover, this linear function f is given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^m$, where $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$.

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- Our next goal is to generalize Theorem 1.10.5 to linear functions $f : U \to V$, where U and V are vector spaces over a field $\mathbb{F}$, and U is **finite-dimensional**.
  - Instead of using the standard basis $\mathcal{E}_m = \{\mathbf{e}_1, \dots, \mathbf{e}_m\}$, we will use an arbitrary basis of U.

- Suppose that V is a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V.

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- By Theorem 3.2.7, every vector of V can be written as linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in a unique way, that is, $\forall \mathbf{v} \in V \; \exists! \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t.

$$\mathbf{v} \quad := \quad \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n,$$

and the *coordinate vector* of $\mathbf{v}$ with respect to the basis $\mathcal{B}$ is defined to be

$$\left[\ \mathbf{v}\ \right]_{\mathcal{B}} \quad := \quad \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

- Suppose that V is a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V.
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$$\left[\ \mathbf{v}\ \right]_{\mathcal{B}} \quad := \quad \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

- As our next proposition shows, $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.
  - It essentially allows us to "translate" vectors of an n-dimensional vector space ($n \neq 0$) into vectors in $\mathbb{F}^n$.

Proposition 4.3.1

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V. Then $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

Proof.

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Proof. We start by proving that $\left[\ \cdot\ \right]_{\mathcal{B}}$ is linear.

1. Fix $\mathbf{x}, \mathbf{y} \in V$. WTS $\left[\ \mathbf{x} + \mathbf{y}\ \right]_{\mathcal{B}} = \left[\ \mathbf{x}\ \right]_{\mathcal{B}} + \left[\ \mathbf{y}\ \right]_{\mathcal{B}}$.

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Then $\mathbf{x} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$ and $\mathbf{y} = \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n$;

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$$\mathbf{x} + \mathbf{y} \quad = \quad (\alpha_1 + \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n + \beta_n)\mathbf{v}_n,$$

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$$\mathbf{x} + \mathbf{y} \;\; = \;\; (\alpha_1 + \beta_1)\mathbf{v}_1 + \dots + (\alpha_n + \beta_n)\mathbf{v}_n,$$

and so $\left[\, \mathbf{x}+\mathbf{y} \, \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 + \beta_1 & \dots & \alpha_n + \beta_n \end{array} \right]^T$.

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$$\mathbf{x} + \mathbf{y} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \dots + (\alpha_n + \beta_n)\mathbf{v}_n,$$

and so $\left[\mathbf{x} + \mathbf{y} \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 + \beta_1 & \dots & \alpha_n + \beta_n \end{array} \right]^T$. We now have that (next slide):

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Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V. Then $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

Proof (continued).

$$\begin{aligned}
\left[\ \mathbf{x} + \mathbf{y}\ \right]_{\mathcal{B}} &= \begin{bmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_n + \beta_n \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \\
&= \left[\ \mathbf{x}\ \right]_{\mathcal{B}} + \left[\ \mathbf{y}\ \right]_{\mathcal{B}}.
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Proof (continued). Similarly (details: Lecture Notes):

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So, $\left[\ \cdot \ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is linear.

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Proof (continued). It remains to show that $\big[\ \cdot\ \big]_{\mathcal{B}}$ is a bijection, i.e. that it is one-to-one and onto $\mathbb{F}^n$.

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Since V and $\mathbb{F}^n$ are both n dimensional, Corollary 4.2.10 guarantees that f is one-to-one iff f is onto $\mathbb{F}^n$.

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Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V. Then $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

Proof (continued). It remains to show that $\left[\ \cdot\ \right]_{\mathcal{B}}$ is a bijection, i.e. that it is one-to-one and onto $\mathbb{F}^n$.

Since V and $\mathbb{F}^n$ are both n dimensional, Corollary 4.2.10 guarantees that f is one-to-one iff f is onto $\mathbb{F}^n$. So, it is enough to show that f is onto $\mathbb{F}^n$.

Fix $\begin{bmatrix} \alpha_1 & \ldots & \alpha_n \end{bmatrix}^T \in \mathbb{F}^n$. Set $\mathbf{v} := \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$. Then $\begin{bmatrix} \mathbf{v} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \ldots & \alpha_n \end{bmatrix}^T$. So, $\left[\ \cdot\ \right]_{\mathcal{B}}$ is onto $\mathbb{F}^n$. This completes the argument. □

- Reminder:

Theorem 1.10.5

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be any vectors in $\mathbb{F}^n$. Then there exists a **unique** linear function $f : \mathbb{F}^m \to \mathbb{F}^n$ that satisfies $f(\mathbf{e}_1) = \mathbf{a}_1, \dots, f(\mathbf{e}_m) = \mathbf{a}_m$, where $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of $\mathbb{F}^m$. Moreover, this linear function f is given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^m$, where $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$.

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Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be any vectors in $\mathbb{F}^n$. Then there exists a **unique** linear function $f : \mathbb{F}^m \to \mathbb{F}^n$ that satisfies $f(\mathbf{e}_1) = \mathbf{a}_1, \dots, f(\mathbf{e}_m) = \mathbf{a}_m$, where $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of $\mathbb{F}^m$. Moreover, this linear function f is given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^m$, where $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$.

- Let's generalize this!

Theorem 4.3.2

Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis of U, and let $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$.[a] Then there exists a unique linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, if the vector space U is non-trivial (i.e. $n \neq 0$), then this unique linear function $f : U \to V$ satisfies the following: for all $\mathbf{u} \in U$, we have that

$$f(\mathbf{u}) \;=\; \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n,$$

where $\left[\, \mathbf{u} \,\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array}\right]^T$. On the other hand, if U is trivial (i.e. $U = \{\mathbf{0}\}$),[b] then $f : U \to V$ is given by $f(\mathbf{0}) = \mathbf{0}$.

---

[a]Here, $\mathbf{v}_1, \dots, \mathbf{v}_n$ are arbitrary vectors in V. They are not necessarily pairwise distinct.

[b]Note that in this case, we have that $n = 0$ and $\mathcal{B} = \emptyset$.

Proof.

Proof. Suppose first that the vector space U is trivial, i.e. $n = 0$ and $U = \{\mathbf{0}\}$.

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Proof. Suppose first that the vector space U is trivial, i.e. $n = 0$ and $U = \{\mathbf{0}\}$. Then the function $f : U \to V$ given by $f(\mathbf{0}) = \mathbf{0}$ is obviously linear, and moreover, it vacuously satisfies $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$ (because $n = 0$, and so both $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are empty lists of vectors). The uniqueness of f follows from Proposition 4.1.6.

Proof. Suppose first that the vector space U is trivial, i.e. $n = 0$ and $U = \{\mathbf{0}\}$. Then the function $f : U \to V$ given by $f(\mathbf{0}) = \mathbf{0}$ is obviously linear, and moreover, it vacuously satisfies $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$ (because $n = 0$, and so both $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are empty lists of vectors). The uniqueness of f follows from Proposition 4.1.6.

From now on, we assume that the vector space U is non-trivial, i.e. that $n \neq 0$.

Proof. Suppose first that the vector space U is trivial, i.e. $n = 0$ and $U = \{\mathbf{0}\}$. Then the function $f : U \to V$ given by $f(\mathbf{0}) = \mathbf{0}$ is obviously linear, and moreover, it vacuously satisfies $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$ (because $n = 0$, and so both $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are empty lists of vectors). The uniqueness of f follows from Proposition 4.1.6.

From now on, we assume that the vector space U is non-trivial, i.e. that $n \neq 0$. We must prove the existence and the uniqueness of the linear function f satisfying the required properties.

Proof (continued). **Existence.** Let $f : U \rightarrow V$ be defined as in the statement of the theorem, i.e. for all $\mathbf{u} \in U$, we set

$$f(\mathbf{u}) = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n,$$

where $\left[\begin{array}{c} \mathbf{u} \end{array} \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array} \right]^T$.

Proof (continued). **Existence.** Let $f : U \to V$ be defined as in the statement of the theorem, i.e. for all $\mathbf{u} \in U$, we set

$$f(\mathbf{u}) = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n,$$

where $\begin{bmatrix} \mathbf{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$. Note that this means that for all $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, we have that

$$f(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n) = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n.$$

Proof (continued). **Existence.** Let $f : U \to V$ be defined as in the statement of the theorem, i.e. for all $\mathbf{u} \in U$, we set

$$f(\mathbf{u}) \;=\; \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n,$$

where $\left[\ \mathbf{u}\ \right]_{\mathcal{B}} = \left[\ \alpha_1 \quad \dots \quad \alpha_n\ \right]^T$. Note that this means that for all $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, we have that

$$f(\alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n) \;=\; \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n.$$

Let us show that f is linear and satisfies $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$.

Proof (continued). **Existence.** Let $f : U \to V$ be defined as in the statement of the theorem, i.e. for all $\mathbf{u} \in U$, we set

$$f(\mathbf{u}) = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n,$$

where $\begin{bmatrix} \mathbf{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$. Note that this means that for all $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, we have that

$$f(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n) = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n.$$

Let us show that f is linear and satisfies $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. For the latter, we note that for all $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned} f(\mathbf{u}_i) &= f(0\mathbf{u}_1 + \cdots + 0\mathbf{u}_{i-1} + 1\mathbf{u}_i + 0\mathbf{u}_{i+1} + \cdots + 0\mathbf{u}_n) \\ &= 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{i-1} + 1\mathbf{v}_i + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_n \\ &= \mathbf{v}_i. \end{aligned}$$

This proves that $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$.

Proof (continued). Let us now show that f is linear. We verify that f satisfies the two axioms from the definition of a linear function.

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1. Fix $\mathbf{x}, \mathbf{y} \in U$. WTS $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$.

Proof (continued). Let us now show that f is linear. We verify that f satisfies the two axioms from the definition of a linear function.

1. Fix $\mathbf{x}, \mathbf{y} \in U$. WTS $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$. Set $\left[\begin{array}{c} \mathbf{x} \end{array} \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array} \right]^T$ and $\left[\begin{array}{c} \mathbf{y} \end{array} \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \beta_1 & \dots & \beta_n \end{array} \right]^T$.

Proof (continued). Let us now show that f is linear. We verify that f satisfies the two axioms from the definition of a linear function.

1. Fix $\mathbf{x}, \mathbf{y} \in U$. WTS $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$. Set $\left[\begin{array}{c} \mathbf{x} \end{array} \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array} \right]^T$ and $\left[\begin{array}{c} \mathbf{y} \end{array} \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \beta_1 & \dots & \beta_n \end{array} \right]^T$. We then have that $\left[\begin{array}{c} \mathbf{x}+\mathbf{y} \end{array} \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1+\beta_1 & \dots & \alpha_n+\beta_n \end{array} \right]^T$,

Proof (continued). Let us now show that f is linear. We verify that f satisfies the two axioms from the definition of a linear function.

1. Fix $\mathbf{x}, \mathbf{y} \in U$. WTS $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$. Set $\left[\begin{array}{c} \mathbf{x} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array}\right]^T$ and $\left[\begin{array}{c} \mathbf{y} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \beta_1 & \dots & \beta_n \end{array}\right]^T$. We then have that $\left[\begin{array}{c} \mathbf{x}+\mathbf{y} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1+\beta_1 & \dots & \alpha_n+\beta_n \end{array}\right]^T$, and we see that

$$\begin{array}{rcl} f(\mathbf{x}+\mathbf{y}) & \stackrel{(*)}{=} & (\alpha_1+\beta_1)\mathbf{v}_1 + \dots + (\alpha_n+\beta_n)\mathbf{v}_n \\ & = & (\alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n) + (\beta_1\mathbf{v}_1 + \dots + \beta_n\mathbf{v}_n) \\ & \stackrel{(**)}{=} & f(\mathbf{x}) + f(\mathbf{y}), \end{array}$$

where both (*) and (**) follow from the construction of f.

Proof (continued). 2. Fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$. WTS $f(\alpha \mathbf{u}) = \alpha f(\mathbf{u})$.

Proof (continued). 2. Fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$. WTS $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$. Set $\left[\begin{array}{c} \mathbf{u} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array}\right]^T$.

Proof (continued). 2. Fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$. WTS $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$.
Set $\left[\begin{array}{c} \mathbf{u} \end{array} \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array} \right]^T$. Then
$\left[\begin{array}{c} \alpha\mathbf{u} \end{array} \right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha\alpha_1 & \dots & \alpha\alpha_n \end{array} \right]^T$,

Proof (continued). 2. Fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$. WTS $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$. Set $\left[\begin{array}{c} \mathbf{u} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array}\right]^T$. Then $\left[\begin{array}{c} \alpha\mathbf{u} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha\alpha_1 & \dots & \alpha\alpha_n \end{array}\right]^T$, and we see that

$$\begin{aligned} f(\alpha\mathbf{u}) &\overset{(*)}{=} (\alpha\alpha_1)\mathbf{v}_1 + \dots + (\alpha\alpha_n)\mathbf{v}_n \\ &= \alpha(\alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n) \\ &\overset{(**)}{=} \alpha f(\mathbf{u}), \end{aligned}$$

where both (*) and (**) follow from the construction of f.

Proof (continued). 2. Fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$. WTS $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$. Set $\begin{bmatrix} \mathbf{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$. Then $\begin{bmatrix} \alpha\mathbf{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha\alpha_1 & \dots & \alpha\alpha_n \end{bmatrix}^T$, and we see that

$$\begin{aligned} f(\alpha\mathbf{u}) &\overset{(*)}{=} (\alpha\alpha_1)\mathbf{v}_1 + \dots + (\alpha\alpha_n)\mathbf{v}_n \\ &= \alpha(\alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n) \\ &\overset{(**)}{=} \alpha f(\mathbf{u}), \end{aligned}$$

where both (*) and (**) follow from the construction of f.

By 1. and 2., we see that f is linear. This completes the proof of existence.

Proof (continued). **Uniqueness.** Let $f_1, f_2 : U \to V$ be linear functions that satisfy $f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n$ and $f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n$. WTS $f_1 = f_2$.

Proof (continued). **Uniqueness.** Let $f_1, f_2 : U \to V$ be linear functions that satisfy $f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n$ and $f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n$. WTS $f_1 = f_2$. Fix $\mathbf{u} \in U$. WTS

$f_1(\mathbf{u}) = f_2(\mathbf{u})$.

Proof (continued). **Uniqueness.** Let $f_1, f_2 : U \to V$ be linear functions that satisfy $f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n$ and $f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n$. WTS $f_1 = f_2$. Fix $\mathbf{u} \in U$. WTS $f_1(\mathbf{u}) = f_2(\mathbf{u})$. Set $\begin{bmatrix} \mathbf{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$.

Proof (continued). **Uniqueness.** Let $f_1, f_2 : U \to V$ be linear functions that satisfy $f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n$ and $f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n$. WTS $f_1 = f_2$. Fix $\mathbf{u} \in U$. WTS $f_1(\mathbf{u}) = f_2(\mathbf{u})$. Set $\left[\, \mathbf{u} \,\right]_{\mathcal{B}} = \left[\, \alpha_1 \ \ \dots \ \ \alpha_n \,\right]^T$. Then

$$
\begin{array}{rcll}
f_1(\mathbf{u}) & = & f_1(\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n) & \\
& = & \alpha_1 f_1(\mathbf{u}_1) + \dots + \alpha_n f_1(\mathbf{u}_n) & \text{by the linearity of } f_1 \text{ (and more precisely, by Proposition 4.1.5)} \\
& = & \alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n & \text{because } f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n \\
& = & \alpha_1 f_2(\mathbf{u}_1) + \dots + \alpha_n f_2(\mathbf{u}_n) & \text{because } f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n \\
& = & f_2(\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n) & \text{by the linearity of } f_2 \text{ (and more precisely, by Proposition 4.1.5)} \\
& = & f_2(\mathbf{u}). &
\end{array}
$$

Thus, $f_1 = f_2$. This proves uniqueness. $\square$

Theorem 4.3.2

Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis of U, and let $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$.[a] Then there exists a unique linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, if the vector space U is non-trivial (i.e. $n \neq 0$), then this unique linear function $f : U \to V$ satisfies the following: for all $\mathbf{u} \in U$, we have that

$$f(\mathbf{u}) = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n,$$

where $\begin{bmatrix} \mathbf{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$. On the other hand, if U is trivial (i.e. $U = \{\mathbf{0}\}$),[b] then $f : U \to V$ is given by $f(\mathbf{0}) = \mathbf{0}$.

---

[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_n$ are arbitrary vectors in V. They are not necessarily pairwise distinct.

[b] Note that in this case, we have that $n = 0$ and $\mathcal{B} = \emptyset$.

Corollary 4.3.3

Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set of vectors in U, and let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$.[a] Then there exists a linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_k) = \mathbf{v}_k$. Moreover, if V is non-trivial, then this linear function f is unique iff $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U.

[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_k$ are arbitrary vectors in V. They are not necessarily pairwise distinct.

- **Remark:** If V is trivial (i.e. $V = \{\mathbf{0}\}$, and consequently $\mathbf{v}_1 = \dots = \mathbf{v}_k = \mathbf{0}$), then there exists exactly one **function** from U to V, this function maps all elements of U to $\mathbf{0}$, and obviously, it is linear.

Corollary 4.3.3

Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set of vectors in U, and let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$.[a] Then there exists a linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_k) = \mathbf{v}_k$. Moreover, if V is non-trivial, then this linear function f is unique iff $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U.

[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_k$ are arbitrary vectors in V. They are not necessarily pairwise distinct.

Proof (outline).

Corollary 4.3.3

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[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_k$ are arbitrary vectors in V. They are not necessarily pairwise distinct.

Proof (outline). Using Theorem 3.2.19, we extend $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ to a basis of U, and then we apply Theorem 4.3.2. The details are left as an exercise. $\square$

3. Isomorphisms

3. Isomorphisms

- Recall that, for vector spaces U and V over a field $\mathbb{F}$, a function $f : U \to V$ is an *isomorphism* if it is linear and a bijection.

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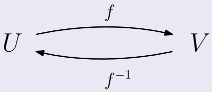
- Recall that, for vector spaces U and V over a field $\mathbb{F}$, a function $f : U \to V$ is an *isomorphism* if it is linear and a bijection.
- Vector spaces U and V (over the same field $\mathbb{F}$) are *isomorphic*, and we write $U \cong V$, if there exits an isomorphism $f : U \to V$.

Proposition 4.4.1

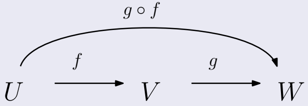
Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be an isomorphism. Then $f^{-1} : V \to U$ is also an isomorphism.



Proof. The same as for isomorphisms $f : \mathbb{F}^n \to \mathbb{F}^n$ (details: Lecture Notes). $\square$

Proposition 4.4.2

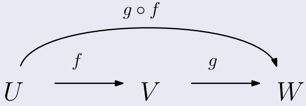
Let U, V, and W be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ and $g : V \to W$ be isomorphisms. Then $g \circ f : U \to W$ is an isomorphism.



Proof.

Proposition 4.4.2

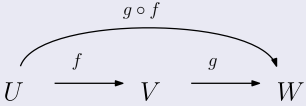
Let U, V, and W be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ and $g : V \to W$ be isomorphisms. Then $g \circ f : U \to W$ is an isomorphism.



Proof. Since $f : U \to V$ and $g : V \to W$ are linear functions (because they are isomorphisms), Proposition 4.1.7 guarantees that their composition $g \circ f : U \to W$ is also linear.

Proposition 4.4.2

Let U, V, and W be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ and $g : V \to W$ be isomorphisms. Then $g \circ f : U \to W$ is an isomorphism.



Proof. Since $f : U \to V$ and $g : V \to W$ are linear functions (because they are isomorphisms), Proposition 4.1.7 guarantees that their composition $g \circ f : U \to W$ is also linear.

Since $f : U \to V$ and $g : V \to W$ are bijections, Proposition 1.10.17 guarantees that $g \circ f : U \to W$ is also a bijection.

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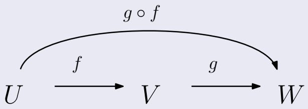

Proof. Since $f : U \to V$ and $g : V \to W$ are linear functions (because they are isomorphisms), Proposition 4.1.7 guarantees that their composition $g \circ f : U \to W$ is also linear.

Since $f : U \to V$ and $g : V \to W$ are bijections, Proposition 1.10.17 guarantees that $g \circ f : U \to W$ is also a bijection.

So, $g \circ f : U \to W$ is linear and a bijection, i.e. it is an isomorphism. $\square$

Theorem 4.4.3

Let U, V, and W be vector spaces over a field $\mathbb{F}$. Then all the following hold:

(a) $U \cong U$;

(b) if $U \cong V$, then $V \cong U$;

(c) if $U \cong V$ and $V \cong W$, then $U \cong W$.

Proof.

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- Reminder: Theorem 4.2.13 (schematically and informally):

$$f : U \overset{\text{linear}}{\longrightarrow} V$$

(a)-(b)	$\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent	$\overset{\text{if } f \text{ is 1-1}}{\Longrightarrow}$ $\underset{\text{always}}{\Longleftarrow}$	$f(\mathbf{u}_1), \ldots, f(\mathbf{u}_k)$ are linearly independent
(c)-(d)	$\mathbf{u}_1, \ldots, \mathbf{u}_k$ span U	$\overset{\text{if } f \text{ is onto}}{\Longrightarrow}$ $\underset{\text{if } f \text{ is 1-1}}{\Longleftarrow}$	$f(\mathbf{u}_1), \ldots, f(\mathbf{u}_k)$ span V

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Theorem 4.4.4

Let U and V be vector spaces over a field $\mathbb{F}$, let $f : U \to V$ be an isomorphism, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$. Then all the following hold:

(a) vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly independent in U iff vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ are linearly independent in V;

(b) vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ span U iff vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span V;

(c) $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U iff $\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\}$ is a basis of V.

Proof. This follows from Theorem 4.2.13 (details: Lecture Notes).

Theorem 4.4.4

Let U and V be vector spaces over a field $\mathbb{F}$, let $f : U \to V$ be an isomorphism, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$. Then all the following hold:

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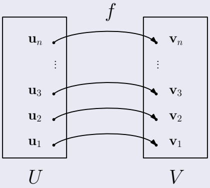
(c) $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U iff $\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\}$ is a basis of V.

- Proposition 4.4.5 (next slide) is a converse of sorts of Theorem 4.4.4(c).
  - It essentially states that any linear function that (injectively) maps a basis onto a basis is an isomorphism.

Proposition 4.4.5

Let U and V be finite-dimensional vector spaces over a field $\mathbb{F}$. Assume that $\dim(U) = \dim(V) =: n$. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for U, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V. Then there exists a unique linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, this linear function f is an isomorphism.



Proof.

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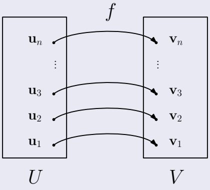

Proof. The existence and uniqueness of the linear function f follows from Theorem 4.3.2.

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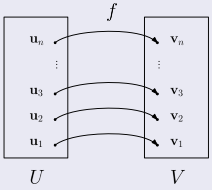
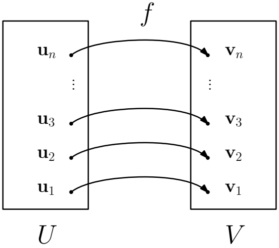
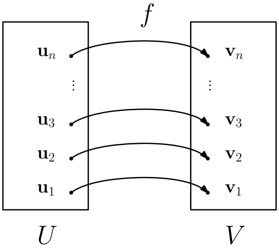


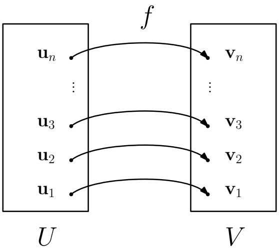
Proof. The existence and uniqueness of the linear function f follows from Theorem 4.3.2. But by hypothesis, U and V are finite-dimensional vector spaces satisfying $\dim(U) = \dim(V)$, and so by Corollary 4.2.10, it is enough to show that f is onto.



Proof (continued). Fix $\mathbf{v} \in V$.



Proof (continued). Fix $\mathbf{v} \in V$. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V, we know that there exist scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$.



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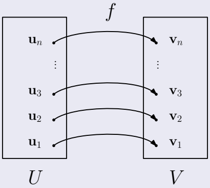
$$\begin{aligned} f(\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n) &\overset{(*)}{=} \alpha_1 f(\mathbf{u}_1) + \dots + \alpha_n f(\mathbf{u}_n) \\ &= \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n \\ &= \mathbf{v}, \end{aligned}$$

where (*) follows from the linearity of f (and more precisely, from Proposition 4.1.5). So, f is onto, and we are done. □

Proposition 4.4.5

Let U and V be finite-dimensional vector spaces over a field $\mathbb{F}$. Assume that $\dim(U) = \dim(V) =: n$. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for U, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V. Then there exists a unique linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, this linear function f is an isomorphism.



- Reminder:

Theorem 4.2.14

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) if f is one-to-one, then $\dim(U) \leq \dim(V)$;

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- By Theorem 4.2.14(c), any two isomorphic vector spaces have the same dimension.

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- By Theorem 4.2.14(c), any two isomorphic vector spaces have the same dimension.
- Theorem 4.4.6 (next slide) guarantees that, in the case of **finite-dimensional** vector spaces, the converse is also true: any two vector spaces (over the same field) that have the same finite dimension are isomorphic.
    - We give two proofs of Theorem 4.4.6!

Theorem 4.4.6

Let U and V be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then U and V are isomorphic iff $\dim(U) = \dim(V)$.

- **Warning:** This theorem is only true for finite-dimensional vector spaces, and it becomes false for infinite-dimensional ones.

Theorem 4.4.6

Let U and V be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then U and V are isomorphic iff $\dim(U) = \dim(V)$.

Proof#1.

Theorem 4.4.6

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Suppose, conversely, that $\dim(U) = \dim(V) =: n$.

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Theorem 4.4.6

Let U and V be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then U and V are isomorphic iff $\dim(U) = \dim(V)$.

Proof#2.

Theorem 4.4.6

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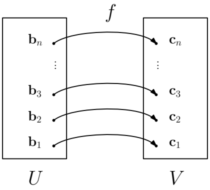

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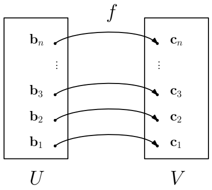


So, U and V are isomorphic. $\square$

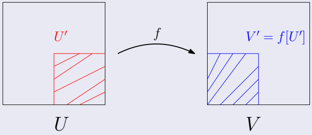
Proposition 4.4.7

Let U and V be a vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be an isomorphism, and let $U' \subseteq U$. Then U' is a subspace of U iff $V' := f[U']$ is a subspace of V. Moreover, in this case, all the following hold:

(a) the function $f' : U' \to V'$ given by $f'(\mathbf{u}) = f(\mathbf{u})$ for all $\mathbf{u} \in U'$ is an isormophism;

(b) $U' \cong V'$;

(c) $\dim(U') = \dim(V')$.



Proof. Lecture Notes. □

4. An application of isomorphisms: transforming polynomials and matrices into vectors

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- Reminder:

Theorem 4.4.6

Let U and V be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then U and V are isomorphic iff $\dim(U) = \dim(V)$.

4. An application of isomorphisms: transforming polynomials and matrices into vectors

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- By Theorem 4.4.6, for all positive integers n and fields $\mathbb{F}$, every n-dimensional vector space V over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.

4. An application of isomorphisms: transforming polynomials and matrices into vectors

- Reminder:

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Let U and V be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then U and V are isomorphic iff $\dim(U) = \dim(V)$.

- By Theorem 4.4.6, for all positive integers n and fields $\mathbb{F}$, every n-dimensional vector space V over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.
- Moreover, by Proposition 4.3.1, given any basis $\mathcal{B}$ of such a vector space V, the coordinate function $[\ \cdot\ ]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

4. An application of isomorphisms: transforming polynomials and matrices into vectors

- Reminder:

Theorem 4.4.6

Let U and V be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then U and V are isomorphic iff $\dim(U) = \dim(V)$.

- By Theorem 4.4.6, for all positive integers n and fields $\mathbb{F}$, every n-dimensional vector space V over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.
- Moreover, by Proposition 4.3.1, given any basis $\mathcal{B}$ of such a vector space V, the coordinate function $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.
  - This is useful because we have developed powerful computational tools for vectors in $\mathbb{F}^n$.
  - By using isomorphisms, we can reduce problems of computing in an arbitrary n-dimensional vector space to problems of computing in $\mathbb{F}^n$, which we know how to do in many cases.

- **Remark:** When working with coordinate vectors, we must always **specify the basis** that we are working with (i.e. with respect to which the coordinate vectors are computed).

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  - For instance, consider the real vector space $\mathbb{P}^2_{\mathbb{R}}$ of all polynomials of degree at most 2 and with coefficients in $\mathbb{R}$.
  - There are two "obvious" bases to chose for $\mathbb{P}^2_{\mathbb{R}}$, namely $\mathcal{A}_1 = \{1, x, x^2\}$ and $\mathcal{A}_2 = \{x^2, x, 1\}$.

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  - For instance, consider the real vector space $\mathbb{P}^2_{\mathbb{R}}$ of all polynomials of degree at most 2 and with coefficients in $\mathbb{R}$.
  - There are two "obvious" bases to chose for $\mathbb{P}^2_{\mathbb{R}}$, namely $\mathcal{A}_1 = \{1, x, x^2\}$ and $\mathcal{A}_2 = \{x^2, x, 1\}$.
  - For a polynomial $p(x) = a_2x^2 + a_1x + a_0$ (with $a_0, a_1, a_2 \in \mathbb{R}$), we have

$$\left[\ p(x)\ \right]_{\mathcal{A}_1} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \qquad \text{and} \qquad \left[\ p(x)\ \right]_{\mathcal{A}_2} = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}.$$

- **Remark:** When working with coordinate vectors, we must always **specify the basis** that we are working with (i.e. with respect to which the coordinate vectors are computed).
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  - For instance, consider the real vector space $\mathbb{P}^2_{\mathbb{R}}$ of all polynomials of degree at most 2 and with coefficients in $\mathbb{R}$.
  - There are two "obvious" bases to chose for $\mathbb{P}^2_{\mathbb{R}}$, namely $\mathcal{A}_1 = \{1, x, x^2\}$ and $\mathcal{A}_2 = \{x^2, x, 1\}$.
  - For a polynomial $p(x) = a_2x^2 + a_1x + a_0$ (with $a_0, a_1, a_2 \in \mathbb{R}$), we have

$$\left[\ p(x)\ \right]_{\mathcal{A}_1} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \qquad \text{and} \qquad \left[\ p(x)\ \right]_{\mathcal{A}_2} = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}.$$

  - As we can see, the coordinate vectors are different (whenever $a_0 \neq a_2$), which is why we have to be careful to specify what basis we are working with.

Proposition 4.4.8

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis of V. Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ ($m \geq 1$) be some vectors in V, and for all $i \in \{1, \dots, n\}$, set $\mathbf{a}_i := \left[\ \mathbf{v}_i\ \right]_{\mathcal{B}}$. Set $A := \left[\ \mathbf{a}_1 \quad \dots \quad \mathbf{a}_m\ \right]$. Then all the following hold:

a) $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a linearly independent set in V iff $\text{rank}(A) = m$ (i.e. A has full column rank);
b) $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a spanning set of V iff $\text{rank}(A) = n$ (i.e. A has full row rank);
c) $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis of V iff $\text{rank}(A) = n = m$ (i.e. A is a square matrix of full rank).

- Proof: Lecture Notes. (Follows easily from results that we have already proven.)

Example 4.4.9

Consider the following sets of polynomials (with coefficients understood to be in $\mathbb{R}$):

(a) $\mathcal{A} = \{x^2 + x, x^3 + 1, x, x^2 + 1\}$;

(b) $\mathcal{B} = \{3x^3 + 2x^2 + x + 1, 6x^3 + 4x^2 + 5x + 6, 5x + 6, 2x + 2\}$;

(c) $\mathcal{C} = \{x^3 + 1, x^3 + x^2, x^2 + x, x + 1, 1, x\}$;

(d) $\mathcal{D} = \{x^3, 2x^2 + 3x, 4x^3 + 5x + 6\}$.

For each of the four sets above, determine whether

- it is linearly independent in $\mathbb{P}^3_{\mathbb{R}}$;
- it spans $\mathbb{P}^3_{\mathbb{R}}$;
- it is a basis of $\mathbb{P}^3_{\mathbb{R}}$.

- We give a solution of (a) and (d). For parts (b) and (d), see the Lecture Notes.

Solution.

Example 4.4.9

Consider the following sets of polynomials (with coefficients understood to be in $\mathbb{R}$):

(a) $\mathcal{A} = \{x^2 + x, x^3 + 1, x, x^2 + 1\}$;

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- it is linearly independent in $\mathbb{P}^3_{\mathbb{R}}$;
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- it is a basis of $\mathbb{P}^3_{\mathbb{R}}$.

- We give a solution of (a) and (d). For parts (b) and (d), see the Lecture Notes.

Solution. In what follows, we will use the basis $\mathcal{P} = \{1, x, x^2, x^3\}$ of $\mathbb{P}^3_{\mathbb{R}}$.

Example 4.4.9

(a) $\mathcal{A} = \{x^2 + x, x^3 + 1, x, x^2 + 1\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

Example 4.4.9

(a) $\mathcal{A} = \{x^2 + x, x^3 + 1, x, x^2 + 1\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

(a) We set

- $\mathbf{a}_1 := \begin{bmatrix} x^2 + x \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$;
- $\mathbf{a}_2 := \begin{bmatrix} x^3 + 1 \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$;
- $\mathbf{a}_3 := \begin{bmatrix} x \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$;
- $\mathbf{a}_4 := \begin{bmatrix} x^2 + 1 \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T$;

Further, we set

$$A := \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Example 4.4.9

- (a) $\mathcal{A} = \{x^2 + x, x^3 + 1, x, x^2 + 1\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

(a) We set

- $\mathbf{a}_1 := \begin{bmatrix} x^2 + x \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$;
- $\mathbf{a}_2 := \begin{bmatrix} x^3 + 1 \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$;
- $\mathbf{a}_3 := \begin{bmatrix} x \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$;
- $\mathbf{a}_4 := \begin{bmatrix} x^2 + 1 \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T$;

Further, we set

$$A := \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

By row reducing, we get that $\text{RREF}(A) = I_4$, and consequently, $\text{rank}(A) = 4$.

Example 4.4.9

(a) $\mathcal{A} = \{x^2 + x, x^3 + 1, x, x^2 + 1\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

(a) We set

- $\mathbf{a}_1 := \left[\ x^2 + x \ \right]_{\mathcal{P}} = \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \end{array} \right]^T$;
- $\mathbf{a}_2 := \left[\ x^3 + 1 \ \right]_{\mathcal{P}} = \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \end{array} \right]^T$;
- $\mathbf{a}_3 := \left[\ x \ \right]_{\mathcal{P}} = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \end{array} \right]^T$;
- $\mathbf{a}_4 := \left[\ x^2 + 1 \ \right]_{\mathcal{P}} = \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \end{array} \right]^T$;

Further, we set

$$A \quad := \quad \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{array} \right] \quad = \quad \left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array} \right].$$

By row reducing, we get that $\text{RREF}(A) = I_4$, and consequently, $\text{rank}(A) = 4$. So, by Proposition 4.4.8, $\mathcal{A}$ is a basis of $\mathbb{P}^3_{\mathbb{R}}$, and in particular, it is a linearly independent set in $\mathbb{P}^3_{\mathbb{R}}$, as well as a spanning set of $\mathbb{P}^3_{\mathbb{R}}$.

Example 4.4.9

(c) $\mathcal{C} = \{x^3+1, x^3+x^2, x^2+x, x+1, 1, x\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

Example 4.4.9

- (c) $\mathcal{C} = \{x^3+1, x^3+x^2, x^2+x, x+1, 1, x\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

(c) We set

- $\mathbf{c}_1 := \left[\; x^3+1 \;\right]_{\mathcal{P}} = \left[\; 1 \quad 0 \quad 0 \quad 1 \;\right]^T$;
- $\mathbf{c}_2 := \left[\; x^3+x^2 \;\right]_{\mathcal{P}} = \left[\; 0 \quad 0 \quad 1 \quad 1 \;\right]^T$;
- $\mathbf{c}_3 := \left[\; x^2+x \;\right]_{\mathcal{P}} = \left[\; 0 \quad 1 \quad 1 \quad 0 \;\right]^T$;
- $\mathbf{c}_4 := \left[\; x+1 \;\right]_{\mathcal{P}} = \left[\; 1 \quad 1 \quad 0 \quad 0 \;\right]^T$;
- $\mathbf{c}_5 := \left[\; 1 \;\right]_{\mathcal{P}} = \left[\; 1 \quad 0 \quad 0 \quad 0 \;\right]^T$;
- $\mathbf{c}_6 := \left[\; x \;\right]_{\mathcal{P}} = \left[\; 0 \quad 1 \quad 0 \quad 0 \;\right]^T$.

Further, we set

$$C \;:=\; \left[\; \mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4 \quad \mathbf{c}_5 \quad \mathbf{c}_6 \;\right] \;=\; \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 4.4.9

- $\mathcal{C} = \{x^3+1, x^3+x^2, x^2+x, x+1, 1, x\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

(c) We set

- $\mathbf{c}_1 := \begin{bmatrix} x^3+1 \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$;
- $\mathbf{c}_2 := \begin{bmatrix} x^3+x^2 \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$;
- $\mathbf{c}_3 := \begin{bmatrix} x^2+x \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$;
- $\mathbf{c}_4 := \begin{bmatrix} x+1 \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$;
- $\mathbf{c}_5 := \begin{bmatrix} 1 \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$;
- $\mathbf{c}_6 := \begin{bmatrix} x \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$.

Further, we set

$$C := \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 & \mathbf{c}_5 & \mathbf{c}_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By row reducing, we get that (next slide):

Example 4.4.9

(c) $\mathcal{C} = \{x^3 + 1, x^3 + x^2, x^2 + x, x + 1, 1, x\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

(c)

$$\text{RREF}(C) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

Example 4.4.9

(c) $\mathcal{C} = \{x^3 + 1, x^3 + x^2, x^2 + x, x + 1, 1, x\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

(c)

$$\text{RREF}(C) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

and consequently, $\text{rank}(C) = 4$.

Example 4.4.9

(c) $\mathcal{C} = \{x^3 + 1, x^3 + x^2, x^2 + x, x + 1, 1, x\}$;

Solution (continued). Reminder: $\mathcal{P} = \{1, x, x^2, x^3\}$.

(c)

$$\text{RREF}(C) \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

and consequently, $\text{rank}(C) = 4$. So, by Proposition 4.4.8, $\mathcal{C}$ is not linearly independent, it is spanning set of $\mathbb{P}^3_{\mathbb{R}}$, and it is not a basis of $\mathbb{P}^3_{\mathbb{R}}$. $\square$

Example 4.4.9

Consider the following sets of polynomials (with coefficients understood to be in $\mathbb{R}$):

(a) $\mathcal{A} = \{x^2 + x, x^3 + 1, x, x^2 + 1\}$;

(b) $\mathcal{B} = \{3x^3 + 2x^2 + x + 1, 6x^3 + 4x^2 + 5x + 6, 5x + 6, 2x + 2\}$;

(c) $\mathcal{C} = \{x^3 + 1, x^3 + x^2, x^2 + x, x + 1, 1, x\}$;

(d) $\mathcal{D} = \{x^3, 2x^2 + 3x, 4x^3 + 5x + 6\}$.

For each of the four sets above, determine whether

- it is linearly independent in $\mathbb{P}^3_{\mathbb{R}}$;
- it spans $\mathbb{P}^3_{\mathbb{R}}$;
- it is a basis of $\mathbb{P}^3_{\mathbb{R}}$.

- See the Lecture Notes for a similar problem, only with matrices instead of polynomials.

Example 4.4.12

Consider the following polynomials in $\mathbb{P}_{\mathbb{Z}_3}$:

- $p_1(x) = x^4 + 2;$
- $p_2(x) = x^3 + x^2;$
- $p_3(x) = x^4 + x^3 + x^2 + 2;$
- $p_4(x) = 2x^4 + x^3 + x^2 + 1;$
- $p_5(x) = 2x + 1.$

Set $U := \text{Span}(p_1(x), \dots, p_5(x))$. Find a basis $\mathcal{B}$ of U. What is $\dim(U)$? For each $i \in \{1, \dots, 5\}$ s.t. $p_i(x)$ is **not** in the basis $\mathcal{B}$, express $p_i(x)$ as a linear combination of the basis vectors in $\mathcal{B}$.

Solution.

Example 4.4.12

Consider the following polynomials in $\mathbb{P}_{\mathbb{Z}_3}$:

- $p_1(x) = x^4 + 2$;
- $p_2(x) = x^3 + x^2$;
- $p_3(x) = x^4 + x^3 + x^2 + 2$;
- $p_4(x) = 2x^4 + x^3 + x^2 + 1$;
- $p_5(x) = 2x + 1$.

Set $U := \text{Span}(p_1(x), \dots, p_5(x))$. Find a basis $\mathcal{B}$ of U. What is $\dim(U)$? For each $i \in \{1, \dots, 5\}$ s.t. $p_i(x)$ is **not** in the basis $\mathcal{B}$, express $p_i(x)$ as a linear combination of the basis vectors in $\mathcal{B}$.

Solution. Note that polynomials $p_1(x), \dots, p_5(x)$ are all of degree at most 4, and they all belong to $\mathbb{P}^4_{\mathbb{Z}_3}$. Thus, $U = \text{Span}(p_1(x), \dots, p_5(x))$ is a subspace of $\mathbb{P}^4_{\mathbb{Z}_3}$. We know that

$$\mathcal{A} \;=\; \{1, x, x^2, x^3, x^4\}$$

is a basis of $\mathbb{P}^4_{\mathbb{Z}_3}$. The coordinate vectors of $p_1(x), \dots, p_5(x)$ with respect to the basis $\mathcal{A}$ are as follows (next slide):

Example 4.4.12

Consider the following polynomials in $\mathbb{P}_{\mathbb{Z}_3}$:

- $p_1(x) = x^4 + 2$;
- $p_2(x) = x^3 + x^2$;
- $p_3(x) = x^4 + x^3 + x^2 + 2$;
- $p_4(x) = 2x^4 + x^3 + x^2 + 1$;
- $p_5(x) = 2x + 1$.

Set $U := \text{Span}\big(p_1(x), \dots, p_5(x)\big)$. Find a basis $\mathcal{B}$ of U. What is $\dim(U)$? For each $i \in \{1, \dots, 5\}$ s.t. $p_i(x)$ is **not** in the basis $\mathcal{B}$, express $p_i(x)$ as a linear combination of the basis vectors in $\mathcal{B}$.

Solution (continued). Reminder: $\mathcal{A} = \{1, x, x^2, x^3, x^4\}$.

- $\left[\ p_1(x)\ \right]_{\mathcal{A}} = \left[\begin{array}{ccccc} 2 & 0 & 0 & 0 & 1 \end{array}\right]^T$;
- $\left[\ p_2(x)\ \right]_{\mathcal{A}} = \left[\begin{array}{ccccc} 0 & 0 & 1 & 1 & 0 \end{array}\right]^T$;
- $\left[\ p_3(x)\ \right]_{\mathcal{A}} = \left[\begin{array}{ccccc} 2 & 0 & 1 & 1 & 1 \end{array}\right]^T$;
- $\left[\ p_4(x)\ \right]_{\mathcal{A}} = \left[\begin{array}{ccccc} 1 & 0 & 1 & 1 & 2 \end{array}\right]^T$;
- $\left[\ p_5(x)\ \right]_{\mathcal{A}} = \left[\begin{array}{ccccc} 1 & 2 & 0 & 0 & 0 \end{array}\right]^T$.

Solution (continued). Reminder: $\mathcal{A} = \{1, x, x^2, x^3, x^4\}$.

We form the matrix

$$A = \left[\ \left[\ p_1(x)\ \right]_{\mathcal{A}} \quad \cdots \quad \left[\ p_5(x)\ \right]_{\mathcal{A}}\ \right] = \begin{bmatrix} 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{bmatrix},$$

and by row reducing, we obtain the following (pivot columns are in red, and non-pivot columns are in blue):

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that the pivot columns of A are its first, second, and fifth column. Therefore, $\mathcal{C} := \left\{ \left[\ p_1(x)\ \right]_{\mathcal{A}}, \left[\ p_2(x)\ \right]_{\mathcal{A}}, \left[\ p_5(x)\ \right]_{\mathcal{A}} \right\}$ is a basis of $\mathrm{Col}(A) = \mathrm{Span}\Big(\left[\ p_1(x)\ \right]_{\mathcal{A}}, \dots, \left[\ p_5(x)\ \right]_{\mathcal{A}} \Big)$.

Consequently, $\mathcal{B} := \Big\{ p_1(x), p_2(x), p_5(x) \Big\}$ is a basis of $U = \mathrm{Span}(p_1(x), \dots, p_5(x))$, and it follows that $\dim(U) = 3$.

Solution (cont.). Reminder: $A = \left[\left[\, p_1(x) \, \right]_{\mathcal{A}} \; \cdots \; \left[\, p_5(x) \, \right]_{\mathcal{A}} \right]$,

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$\mathcal{B} := \left\{ p_1(x), p_2(x), p_5(x) \right\}$ is a basis of $U = \text{Span}\big(p_1(x), \ldots, p_5(x)\big)$.

Solution (cont.). Reminder: $A = \left[\ \left[\ p_1(x)\ \right]_{\mathcal{A}} \quad \cdots \quad \left[\ p_5(x)\ \right]_{\mathcal{A}}\ \right]$,

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$\mathcal{B} := \left\{ p_1(x), p_2(x), p_5(x) \right\}$ is a basis of $U = \text{Span}\big(p_1(x), \ldots, p_5(x)\big)$.

It remains to express $p_3(x)$ and $p_4(x)$ as a linear combination of the vectors (polynomials) in $\mathcal{B}$. First, we have that

- $\left[\ p_3(x)\ \right]_{\mathcal{A}} \overset{(*)}{=} \left[\ p_1(x)\ \right]_{\mathcal{A}} + \left[\ p_2(x)\ \right]_{\mathcal{A}} \overset{(**)}{=} \left[\ p_1(x) + p_2(x)\ \right]_{\mathcal{A}}$,
- $\left[\ p_4(x)\ \right]_{\mathcal{A}} \overset{(*)}{=} 2\left[\ p_1(x)\ \right]_{\mathcal{A}} + \left[\ p_2(x)\ \right]_{\mathcal{A}} \overset{(**)}{=} \left[\ 2p_1(x) + p_2(x)\ \right]_{\mathcal{A}}$,

where both instances of (*) were obtained from the matrix RREF(A), and both instances of (**) follow from the fact that $\left[\ \cdot\ \right]_{\mathcal{A}} : \mathbb{P}^4_{\mathbb{Z}_3} \to \mathbb{Z}_3^5$ is linear (because it is an isomorphism). Since $\left[\ \cdot\ \right]_{\mathcal{A}}$ is also one-to-one (again, because it is an isomorphism), we get that

- $p_3(x) = p_1(x) + p_2(x)$,
- $p_4(x) = 2p_1(x) + p_2(x)$,

and we are done. $\square$

Example 4.4.14

Consider the following polynomials in $\mathbb{P}^3_{\mathbb{Z}_3}$:

- $p_1(x) = x^3 + 1$;
- $p_2(x) = 2x^3 + 2$;
- $p_3(x) = x^2 + 2x + 1$
- $p_4(x) = 2x^3 + x^2 + 2x$.

Find a basis $\mathcal{B}_U$ of $U := \text{Span}\big(p_1(x), p_2(x), p_3(x), p_4(x)\big)$, extend it to a basis $\mathcal{B}$ of $\mathbb{P}^3_{\mathbb{Z}_3}$, and for each $i \in \{1, 2, 3, 4\}$ s.t. $p_i(x) \notin \mathcal{B}$, express $p_i(x)$ as a linear combination of the basis vectors in $\mathcal{B}$.

Solution.

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Solution. We know that $\mathcal{A} := \{1, x, x^2, x^3\}$ is a basis of $\mathbb{P}^3_{\mathbb{Z}_3}$,

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Solution. We know that $\mathcal{A} := \{1, x, x^2, x^3\}$ is a basis of $\mathbb{P}^3_{\mathbb{Z}_3}$, and we let V be the image of U under the isomorphism $[\ \cdot\ ]_{\mathcal{A}}$.

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- $p_2(x) = 2x^3 + 2$;
- $p_3(x) = x^2 + 2x + 1$
- $p_4(x) = 2x^3 + x^2 + 2x$.

Find a basis $\mathcal{B}_U$ of $U := \text{Span}(p_1(x), p_2(x), p_3(x), p_4(x))$, extend it to a basis $\mathcal{B}$ of $\mathbb{P}^3_{\mathbb{Z}_3}$, and for each $i \in \{1, 2, 3, 4\}$ s.t. $p_i(x) \notin \mathcal{B}$, express $p_i(x)$ as a linear combination of the basis vectors in $\mathcal{B}$.

Solution. We know that $\mathcal{A} := \{1, x, x^2, x^3\}$ is a basis of $\mathbb{P}^3_{\mathbb{Z}_3}$, and we let V be the image of U under the isomorphism $\left[\ \cdot\ \right]_{\mathcal{A}}$. Further, we consider the standard basis

$$\mathcal{E}_4 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} = \Big\{ \left[\ 1\ \right]_{\mathcal{A}}, \left[\ x\ \right]_{\mathcal{A}}, \left[\ x^2\ \right]_{\mathcal{A}}, \left[\ x^3\ \right]_{\mathcal{A}} \Big\}$$

of $\mathbb{Z}_3^4$.

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Consider the following polynomials in $\mathbb{P}^3_{\mathbb{Z}_3}$:

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Find a basis $\mathcal{B}_U$ of $U := \text{Span}\big(p_1(x), p_2(x), p_3(x), p_4(x)\big)$, extend it to a basis $\mathcal{B}$ of $\mathbb{P}^3_{\mathbb{Z}_3}$, and for each $i \in \{1, 2, 3, 4\}$ s.t. $p_i(x) \notin \mathcal{B}$, express $p_i(x)$ as a linear combination of the basis vectors in $\mathcal{B}$.

Solution. We know that $\mathcal{A} := \{1, x, x^2, x^3\}$ is a basis of $\mathbb{P}^3_{\mathbb{Z}_3}$, and we let V be the image of U under the isomorphism $\left[\ \cdot\ \right]_{\mathcal{A}}$. Further, we consider the standard basis

$$\mathcal{E}_4 \quad = \quad \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} \quad = \quad \Big\{ \left[\ 1\ \right]_{\mathcal{A}}, \left[\ x\ \right]_{\mathcal{A}}, \left[\ x^2\ \right]_{\mathcal{A}}, \left[\ x^3\ \right]_{\mathcal{A}} \Big\}$$

of $\mathbb{Z}_3^4$. We now form the 4×8 matrix C whose columns are the coordinate vectors of the polynomials

$$p_1(x), \quad p_2(x), \quad p_3(x), \quad p_4(x), \quad 1, \quad x, \quad x^2, \quad x^3$$

with respect to the basis $\mathcal{A}$.

Solution (continued). Here is the matrix C explicitly (with tiny font and dots so that it would fit on the page):

$$C := \left[\begin{array}{ccc|cccc} \left[p_1(x) \right]_{\mathcal{A}} & \cdots & \left[p_4(x) \right]_{\mathcal{A}} & \left[1 \right]_{\mathcal{A}} & \left[x \right]_{\mathcal{A}} & \left[x^2 \right]_{\mathcal{A}} & \left[x^3 \right]_{\mathcal{A}} \end{array} \right].$$

Solution (continued). Here is the matrix C explicitly (with tiny font and dots so that it would fit on the page):

$$C := \left[\begin{array}{ccc|cccc} \left[\, p_1(x) \,\right]_{\mathcal{A}} & \cdots & \left[\, p_4(x) \,\right]_{\mathcal{A}} & \left[\, 1 \,\right]_{\mathcal{A}} & \left[\, x \,\right]_{\mathcal{A}} & \left[\, x^2 \,\right]_{\mathcal{A}} & \left[\, x^3 \,\right]_{\mathcal{A}} \end{array}\right].$$

We then have that

$$C = \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right].$$

By row reducing, we obtain

$$\mathrm{RREF}(C) = \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right],$$

and we see that the pivot columns of C are its first, third, fifth, and sixth column.

Solution (continued). Here is the matrix C explicitly (with tiny font and dots so that it would fit on the page):

$$C := \left[\begin{array}{ccc:cccc} [p_1(x)]_{\mathcal{A}} & \cdots & [p_4(x)]_{\mathcal{A}} & [1]_{\mathcal{A}} & [x]_{\mathcal{A}} & [x^2]_{\mathcal{A}} & [x^3]_{\mathcal{A}} \end{array} \right].$$

We then have that

$$C = \left[\begin{array}{cccc:cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 \end{array} \right].$$

By row reducing, we obtain

$$\mathrm{RREF}(C) = \left[\begin{array}{cccc:cccc} 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array} \right],$$

and we see that the pivot columns of C are its first, third, fifth, and sixth column. By Proposition 3.3.23, the pivot columns of C to the left of the vertical dotted line form a basis of V, and all the pivot columns of C together form a basis of $\mathbb{Z}_3^4$.

Solution (continued). Reminder:

$$\text{RREF}(C) = \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right],$$

So, $\left\{ \left[\ p_1(x)\ \right]_{\mathcal{A}}, \left[\ p_3(x)\ \right]_{\mathcal{A}} \right\}$ is a basis of V, whereas $\left\{ \left[\ p_1(x)\ \right]_{\mathcal{A}}, \left[\ p_3(x)\ \right]_{\mathcal{A}}, \left[\ 1\ \right]_{\mathcal{A}}, \left[\ x\ \right]_{\mathcal{A}} \right\}$ is a basis of $\mathbb{Z}_3^4$ that extends our basis of V.

Solution (continued). Reminder:

$$\mathrm{RREF}(C) \quad = \quad \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right],$$

So, $\left\{ \left[\ p_1(x)\ \right]_{\mathcal{A}}, \left[\ p_3(x)\ \right]_{\mathcal{A}} \right\}$ is a basis of V, whereas $\left\{ \left[\ p_1(x)\ \right]_{\mathcal{A}}, \left[\ p_3(x)\ \right]_{\mathcal{A}}, \left[\ 1\ \right]_{\mathcal{A}}, \left[\ x\ \right]_{\mathcal{A}} \right\}$ is a basis of $\mathbb{Z}_3^4$ that extends our basis of V. Since $\left[\ \cdot\ \right]_{\mathcal{A}}$ is an isomorphism, we see that

$$\mathcal{B}_U \quad := \quad \left\{ p_1(x), p_3(x) \right\}$$

is a basis of U, and that

$$\mathcal{B} \quad := \quad \left\{ p_1(x), p_3(x), 1, x \right\}$$

is a basis of $\mathbb{P}^3_{\mathbb{Z}_3}$ that extends our basis $\mathcal{B}_U$ of U.

Solution (continued). Reminder:

$$\mathrm{RREF}(C) = \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right],$$

So, $\left\{ \left[\ p_1(x)\ \right]_{\mathcal{A}}, \left[\ p_3(x)\ \right]_{\mathcal{A}} \right\}$ is a basis of V, whereas $\left\{ \left[\ p_1(x)\ \right]_{\mathcal{A}}, \left[\ p_3(x)\ \right]_{\mathcal{A}}, \left[\ 1\ \right]_{\mathcal{A}}, \left[\ x\ \right]_{\mathcal{A}} \right\}$ is a basis of $\mathbb{Z}_3^4$ that extends our basis of V. Since $\left[\ \cdot\ \right]_{\mathcal{A}}$ is an isomorphism, we see that

$$\mathcal{B}_U := \left\{ p_1(x), p_3(x) \right\}$$

is a basis of U, and that

$$\mathcal{B} := \left\{ p_1(x), p_3(x), 1, x \right\}$$

is a basis of $\mathbb{P}^3_{\mathbb{Z}_3}$ that extends our basis $\mathcal{B}_U$ of U. Finally, we can read off from $\mathrm{RREF}(C)$ that

- $p_2(x) = 2p_1(x)$,
- $p_4(x) = 2p_1(x) + p_3(x)$,

and we are done. □