Linear Algebra 1

Lecture #9

The column space, row space, and null space of a matrix. The rank of a matrix revisited

Irena Penev

December 11, 2024

- This lecture has five parts:

- This lecture has five parts:
  1. The column space and row space of a matrix (and their relationship with rank)

- This lecture has five parts:
    1. The column space and row space of a matrix (and their relationship with rank)
    2. Matrices of full rank

- This lecture has five parts:
    1. The column space and row space of a matrix (and their relationship with rank)
    2. Matrices of full rank
    3. The rank of matrix products. Left and right inverses of a matrix

- This lecture has five parts:
  1. The column space and row space of a matrix (and their relationship with rank)
  2. Matrices of full rank
  3. The rank of matrix products. Left and right inverses of a matrix
  4. The null space of a matrix

- This lecture has five parts:
  1. The column space and row space of a matrix (and their relationship with rank)
  2. Matrices of full rank
  3. The rank of matrix products. Left and right inverses of a matrix
  4. The null space of a matrix
  5. The Invertible Matrix Theorem (version 2)

1. The column space and row space of a matrix (and their relationship with rank)

1. The column space and row space of a matrix (and their relationship with rank)

- Reminder:

Theorem 3.1.11

Let V be a vector space over a field $\mathbb{F}$, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$ $(k \geq 0)$. Then all the following hold:

(a) $\mathbf{u}_1, \dots, \mathbf{u}_k \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$;

(b) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of V;

(c) for all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of U;

(d) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is precisely the intersection of all subspaces of V that contain the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$.

Definition

For a field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n \times m}$, we define the following:

- the *column space* of A, denoted by $\mathrm{Col}(A)$, is the subspace of $\mathbb{F}^n$ spanned by the columns of A;[a]
- the *row space* of A, denoted by $\mathrm{Row}(A)$, is the subspace of $\mathbb{F}^{1 \times m}$ spanned by the rows of A.[b]

---

[a] More precisely, if $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$, then $\mathrm{Col}(A) := \mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$. The fact that $\mathrm{Col}(A)$ really is a subspace of $\mathbb{F}^n$ follows from Theorem 3.1.11.

[b] More precisely, if $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \end{bmatrix}$ (i.e. $\mathbf{r}_1, \dots, \mathbf{r}_n$ are the rows of A, appearing in A in that order, from top to bottom), then $\mathrm{Row}(A) := \mathrm{Span}(\mathbf{r}_1, \dots, \mathbf{r}_n)$. The fact that $\mathrm{Row}(A)$ really is a subspace of $\mathbb{F}^{1 \times m}$ follows from Theorem 3.1.11.

Proposition 3.3.2

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ be a matrix. Then both the following hold:

(a) $\text{Col}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\}$;

(b) $\text{Row}(A) = \{\mathbf{x}A \mid \mathbf{x} \in \mathbb{F}^{1 \times n}\}$.[a]

---

[a] Note that in the expression $\mathbf{x}A$, we have that $\mathbf{x}$ is a **row** vector with n entries.

Proof.

Proposition 3.3.2

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ be a matrix. Then both the following hold:

(a) $\text{Col}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\}$;

(b) $\text{Row}(A) = \{\mathbf{x}A \mid \mathbf{x} \in \mathbb{F}^{1 \times n}\}$.[a]

[a] Note that in the expression $\mathbf{x}A$, we have that $\mathbf{x}$ is a **row** vector with n entries.

Proof. For (a), we set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$, and we observe that

$$\begin{array}{rcll} \text{Col}(A) & = & \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) & \text{by the definition of Col}(A) \\ & = & \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\} & \text{by Proposition 1.4.4.} \end{array}$$

Proposition 3.3.2

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ be a matrix. Then both the following hold:

(a) $\text{Col}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\}$;

(b) $\text{Row}(A) = \{\mathbf{x}A \mid \mathbf{x} \in \mathbb{F}^{1 \times n}\}$.[a]

[a] Note that in the expression $\mathbf{x}A$, we have that $\mathbf{x}$ is a **row** vector with n entries.

Proof. For (a), we set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$, and we observe that

$$\begin{array}{rcll} \text{Col}(A) & = & \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) & \text{by the definition of Col}(A) \\ & = & \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\} & \text{by Proposition 1.4.4.} \end{array}$$

The proof of (b) is in the Lecture Notes (easy). □

- Our first goal is to prove the following two theorems, which give a recipe for finding bases of the column space and row space of a matrix.

- Our first goal is to prove the following two theorems, which give a recipe for finding bases of the column space and row space of a matrix.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then the pivot columns of A form a basis of $\text{Col}(A)$. Moreover, $\dim(\text{Col}(A)) = \text{rank}(A)$.

- Our first goal is to prove the following two theorems, which give a recipe for finding bases of the column space and row space of a matrix.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then the pivot columns of A form a basis of $\mathrm{Col}(A)$. Moreover, $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$, and let U be any matrix in row echelon form that is row equivalent to A.[a] Then the non-zero rows of U form a basis of $\mathrm{Row}(A)$. Moreover, $\dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$.

[a] It may be that $U = \mathrm{RREF}(A)$, but this assumption is not necessary. U may be any matrix in row echelon form obtained from A via a sequence of elementary row operations. For instance, U may be the matrix obtained from A by performing only the "forward" part of the row reduction algorithm in order to transform A into a matrix in row echelon form.

- Our first goal is to prove the following two theorems, which give a recipe for finding bases of the column space and row space of a matrix.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\text{Col}(A)$. Moreover, $\dim(\text{Col}(A)) = \text{rank}(A)$.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$, and let U be any matrix in row echelon form that is row equivalent to A.[a] Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

[a] It may be that $U = \text{RREF}(A)$, but this assumption is not necessary. U may be any matrix in row echelon form obtained from A via a sequence of elementary row operations. For instance, U may be the matrix obtained from A by performing only the "forward" part of the row reduction algorithm in order to transform A into a matrix in row echelon form.

- First an example, then the proofs (or rather: proof outlines).

Example 3.3.10

Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 2 & 1 \end{bmatrix}$$

with entries understood to be in $\mathbb{R}$.

(a) Compute $\operatorname{rank}(A)$.

(b) Find a basis of $\operatorname{Col}(A)$.

(c) Find a basis of $\operatorname{Row}(A)$.

Solution.

Example 3.3.10

Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 2 & 1 \end{bmatrix}$$

with entries understood to be in $\mathbb{R}$.

(a) Compute $\operatorname{rank}(A)$.

(b) Find a basis of $\operatorname{Col}(A)$.

(c) Find a basis of $\operatorname{Row}(A)$.

Solution. By performing the "forward" part of the row reduction algorithm, we see that the following matrix is a row echelon form of A:

$$U = \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Compute $\operatorname{rank}(A)$.
2. Find a basis of $\operatorname{Col}(A)$.
3. Find a basis of $\operatorname{Row}(A)$.

Solution (continued). Reminder:

$$\underbrace{\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 2 & 1 \end{bmatrix}}_{=A} \sim \underbrace{\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=U}$$

(a) The matrix U has three pivot columns, and so $\operatorname{rank}(A) = 3$.

(b) Find a basis of Col(A).

Solution (continued). Reminder:

$$\underbrace{\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 2 & 1 \end{bmatrix}}_{=A} \sim \underbrace{\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=U}$$

(b) The pivot columns of U are its first, second, and fifth column. So, the pivot columns of A are its first, second, and fifth column, and so those columns of A form a basis of Col(A). More precisely, the following is a basis of Col(A):

$$\left\{ \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ -9 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 6 \\ 2 \end{bmatrix} \right\}.$$

(c) Find a basis of Row(A).

Solution (continued). Reminder:

$$\underbrace{\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 2 & 1 \end{bmatrix}}_{=A} \sim \underbrace{\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=U}$$

(c) The non-zero rows of U form a basis of Row(A). So, the following is a basis of Row(A):

$$\left\{ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}, \begin{bmatrix} 0 & 2 & -4 & 4 & 2 & -6 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \right\}.$$

$\square$

- Let's outline the proof of Theorem 3.3.4.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\text{Col}(A)$. Moreover, $\dim(\text{Col}(A)) = \text{rank}(A)$.

- Our proof of Theorem 3.3.4 relies on the following proposition:

- Let's outline the proof of Theorem 3.3.4.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\mathrm{Col}(A)$. Moreover, $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.

- Our proof of Theorem 3.3.4 relies on the following proposition:

Proposition 3.3.3

Let $\mathbb{F}$ be a field, let $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{F}^n$, and let $B \in \mathbb{F}^{n \times n}$ be an invertible matrix. Then both the following hold:

(a) $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is linearly independent iff $\{B\mathbf{a}_1, \dots, B\mathbf{a}_k\}$ is linearly independent;

(b) $\forall \mathbf{v} \in \mathbb{F}^n$: $\mathbf{v} \in \mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_k)$ iff $B\mathbf{v} \in \mathrm{Span}(B\mathbf{a}_1, \dots, B\mathbf{a}_k)$;

- Proof: Lecture Notes.
  - Easy! Just use the appropriate definitions.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\mathrm{Col}(A)$. Moreover, $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.

Proof (outline).

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then the pivot columns of A form a basis of $\mathrm{Col}(A)$. Moreover, $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.

Proof (outline). Since $r := \mathrm{rank}(A)$ is equal to the number of pivot columns of A the first statement implies the second.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\mathrm{Col}(A)$. Moreover, $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.

Proof (outline). Since $r := \mathrm{rank}(A)$ is equal to the number of pivot columns of A the first statement implies the second.

It remains to prove the first statement. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then the pivot columns of A form a basis of $\mathrm{Col}(A)$. Moreover, $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.

Proof (outline). Since $r := \mathrm{rank}(A)$ is equal to the number of pivot columns of A the first statement implies the second.

It remains to prove the first statement. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$. Let $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ (with $1 \leq i_1 < \cdots < i_r \leq m$) be the pivot columns of A. WTS $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}\}$ is a basis of $\mathrm{Col}(A)$.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then the pivot columns of A form a basis of $\mathrm{Col}(A)$. Moreover, $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.

Proof (outline). Since $r := \mathrm{rank}(A)$ is equal to the number of pivot columns of A the first statement implies the second.

It remains to prove the first statement. Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array}\right]$. Let $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ (with $1 \leq i_1 < \dots < i_r \leq m$) be the pivot columns of A. WTS $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}\}$ is a basis of $\mathrm{Col}(A)$.

Set $U := \mathrm{RREF}(A)$.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\text{Col}(A)$. Moreover, $\dim(\text{Col}(A)) = \text{rank}(A)$.

Proof (outline). Since $r := \text{rank}(A)$ is equal to the number of pivot columns of A the first statement implies the second.

It remains to prove the first statement. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$. Let $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ (with $1 \le i_1 < \dots < i_r \le m$) be the pivot columns of A. WTS $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}\}$ is a basis of $\text{Col}(A)$.

Set $U := \text{RREF}(A)$. Then $A \sim U$, and so by Theorem 1.11.13, there exists an invertible matrix $B \in \mathbb{F}^{n \times n}$ s.t. $U = BA = \begin{bmatrix} B\mathbf{a}_1 & \dots & B\mathbf{a}_m \end{bmatrix}$.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF RREF

Proof (outline, cont.). Reminder: $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$; $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ are the pivot col.'s of A; $\text{RREF}(A) = U = BA = \begin{bmatrix} B\mathbf{a}_1 & \dots & B\mathbf{a}_m \end{bmatrix}$ (B is invertible). WTS $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}\}$ is a basis of $\text{Col}(A)$.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF RREF

Proof (outline, cont.). Reminder: $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$; $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ are the pivot col.'s of A; $\text{RREF}(A) = U = BA = \begin{bmatrix} B\mathbf{a}_1 & \dots & B\mathbf{a}_m \end{bmatrix}$ (B is invertible). WTS $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}\}$ is a basis of $\text{Col}(A)$.

But now since $U = \text{RREF}(A)$, we see that all the following hold:

(i) $B\mathbf{a}_{i_1}, \dots, B\mathbf{a}_{i_r}$ are the pivot columns of U;

(ii) for all $j \in \{1, \dots, r\}$, we have that $B\mathbf{a}_{i_j} = \mathbf{e}_j^n$;

(iii) in any column of U, only the top r entries may possibly be non-zero (the other entries are all zero).

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF RREF

Proof (outline, cont.). Reminder: $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$; $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ are the pivot col.'s of A; $\text{RREF}(A) = U = BA = \begin{bmatrix} B\mathbf{a}_1 & \dots & B\mathbf{a}_m \end{bmatrix}$ (B is invertible). WTS $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}\}$ is a basis of $\text{Col}(A)$.

But now since $U = \text{RREF}(A)$, we see that all the following hold:

(i) $B\mathbf{a}_{i_1}, \dots, B\mathbf{a}_{i_r}$ are the pivot columns of U;

(ii) for all $j \in \{1, \dots, r\}$, we have that $B\mathbf{a}_{i_j} = \mathbf{e}_j^n$;

(iii) in any column of U, only the top r entries may possibly be non-zero (the other entries are all zero).

So, $\{\mathbf{e}_1^n, \dots, \mathbf{e}_r^n\} = \{B\mathbf{a}_{i_1}, \dots, B\mathbf{a}_{i_r}\}$ is a basis of $\text{Col}(U) = \text{Span}(B\mathbf{a}_1, \dots, B\mathbf{a}_m)$.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF RREF

Proof (outline, cont.). Reminder: $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$; $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ are the pivot col.'s of A; $\text{RREF}(A) = U = BA = \begin{bmatrix} B\mathbf{a}_1 & \dots & B\mathbf{a}_m \end{bmatrix}$ (B is invertible). WTS $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}\}$ is a basis of $\text{Col}(A)$.

But now since $U = \text{RREF}(A)$, we see that all the following hold:

(i) $B\mathbf{a}_{i_1}, \dots, B\mathbf{a}_{i_r}$ are the pivot columns of U;

(ii) for all $j \in \{1, \dots, r\}$, we have that $B\mathbf{a}_{i_j} = \mathbf{e}_j^n$;

(iii) in any column of U, only the top r entries may possibly be non-zero (the other entries are all zero).

So, $\{\mathbf{e}_1^n, \dots, \mathbf{e}_r^n\} = \{B\mathbf{a}_{i_1}, \dots, B\mathbf{a}_{i_r}\}$ is a basis of $\text{Col}(U) = \text{Span}(B\mathbf{a}_1, \dots, B\mathbf{a}_m)$. But now Proposition 3.3.3 implies that $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}\}$ is a basis of $\text{Col}(A) = \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$ (full details: Lecture Notes). $\square$

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\mathrm{Col}(A)$. Moreover, $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\text{Col}(A)$. Moreover, $\dim(\text{Col}(A)) = \text{rank}(A)$.

- Now let's outline the proof of Theorem 3.3.9.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$, and let U be any matrix in row echelon form that is row equivalent to A.[a] Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

---

[a] It may be that $U = \text{RREF}(A)$, but this assumption is not necessary. U may be any matrix in row echelon form obtained from A via a sequence of elementary row operations. For instance, U may be the matrix obtained from A by performing only the "forward" part of the row reduction algorithm in order to transform A into a matrix in row echelon form.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n \times m}$ have the same row space.

Proof (outline).

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Proof (outline). By definition, two matrices are row equivalent iff one can be obtained from the other via a sequence of elementary row operations.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Proof (outline). By definition, two matrices are row equivalent iff one can be obtained from the other via a sequence of elementary row operations. So, it is enough to prove that applying one elementary row operation to a matrix does not alter the row space.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Proof (outline). By definition, two matrices are row equivalent iff one can be obtained from the other via a sequence of elementary row operations. So, it is enough to prove that applying one elementary row operation to a matrix does not alter the row space. More precisely, it is enough to prove the following claim.

> ***Claim.*** *Let $A, B \in \mathbb{F}^{n\times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Claim. *Let $A, B \in \mathbb{F}^{n \times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline). Set $A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix}$ (so, $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the rows of A appearing in that order in A, from top to bottom, and similar for B).

Claim. *Let $A, B \in \mathbb{F}^{n \times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline). Set $A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix}$ (so, $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the rows of A appearing in that order in A, from top to bottom, and similar for B). By definition, $\text{Row}(A) = \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ and $\text{Row}(B) = \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n)$.

Claim. *Let $A, B \in \mathbb{F}^{n\times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline). Set $A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix}$ (so, $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the rows of A appearing in that order in A, from top to bottom, and similar for B). By definition, $\text{Row}(A) = \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ and $\text{Row}(B) = \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n)$.

Since B is obtained from A by performing one elementary row operation R, we know that A can be obtained from B by performing one elementary row operation (the one that "undoes" R).

Claim. *Let $A, B \in \mathbb{F}^{n\times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then Row$(A) =$ Row(B).*

Proof of the Claim (outline). Set $A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix}$ (so, $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the rows of A appearing in that order in A, from top to bottom, and similar for B). By definition, $\text{Row}(A) = \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ and $\text{Row}(B) = \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n)$.

Since B is obtained from A by performing one elementary row operation R, we know that A can be obtained from B by performing one elementary row operation (the one that "undoes" R). So, it is enough to show that $\text{Row}(A) \subseteq \text{Row}(B)$, for then an analogous argument will establish that $\text{Row}(B) \subseteq \text{Row}(A)$, and then the result will follow.

Claim. *Let $A, B \in \mathbb{F}^{n \times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). There are three types of elementary row operations: swapping two rows; multiplying a row by a non-zero scalar; adding a scalar multiple of one row to another.

Claim. *Let $A, B \in \mathbb{F}^{n\times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). There are three types of elementary row operations: swapping two rows; multiplying a row by a non-zero scalar; adding a scalar multiple of one row to another. We consider only the third type (the other two types are easier; see the Lecture Notes).

Claim. *Let $A, B \in \mathbb{F}^{n\times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). There are three types of elementary row operations: swapping two rows; multiplying a row by a non-zero scalar; adding a scalar multiple of one row to another. We consider only the third type (the other two types are easier; see the Lecture Notes).

So, suppose that B is obtained from A by adding a scalar multiple of one row to another row.

Claim. *Let $A, B \in \mathbb{F}^{n \times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). There are three types of elementary row operations: swapping two rows; multiplying a row by a non-zero scalar; adding a scalar multiple of one row to another. We consider only the third type (the other two types are easier; see the Lecture Notes).

So, suppose that B is obtained from A by adding a scalar multiple of one row to another row. Then there exist distinct indices $i, j \in \{1, \dots, n\}$ and a scalar $\alpha \in \mathbb{F}$ such that $\mathbf{b}_j = \mathbf{a}_j + \alpha \mathbf{a}_i$, and $\mathbf{b}_k = \mathbf{a}_k$ for all $k \in \{1, \dots, n\} \setminus \{j\}$.

- So, we applied the elementary row operation "$R_j \to R_j + \alpha R_i$."

Claim. *Let $A, B \in \mathbb{F}^{n\times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). Reminder: $B \overset{R_j \to R_j + \alpha R_i}{\sim} A$.
WTS $\text{Row}(A) \subseteq \text{Row}(B)$.

Claim. *Let $A, B \in \mathbb{F}^{n\times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). Reminder: $B \overset{R_j \to R_j + \alpha R_i}{\sim} A$. WTS $\text{Row}(A) \subseteq \text{Row}(B)$.

Now, fix $\mathbf{v} \in \text{Row}(A)$.

Claim. *Let $A, B \in \mathbb{F}^{n \times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). Reminder: $B \overset{R_j \to R_j + \alpha R_i}{\sim} A$.
WTS $\text{Row}(A) \subseteq \text{Row}(B)$.

Now, fix $\mathbf{v} \in \text{Row}(A)$. Then $\exists \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t.
$\mathbf{v} = \alpha_1 \mathbf{a}_1 + \dots + \alpha_n \mathbf{a}_n$.

Claim. *Let $A, B \in \mathbb{F}^{n \times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). Reminder: $B \overset{R_j \to R_j + \alpha R_i}{\sim} A$. WTS $\text{Row}(A) \subseteq \text{Row}(B)$.

Now, fix $\mathbf{v} \in \text{Row}(A)$. Then $\exists \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{v} = \alpha_1 \mathbf{a}_1 + \dots + \alpha_n \mathbf{a}_n$. We now set $\beta_i := \alpha_i - \alpha_j \alpha$, and we set $\beta_k := \alpha_k$ for all $k \in \{1, \dots, n\} \setminus \{i\}$. Then

$$\beta_i \mathbf{b}_i + \beta_j \mathbf{b}_j \quad = \quad (\alpha_i - \alpha_j \alpha)\mathbf{a}_i + \alpha_j(\mathbf{a}_j + \alpha \mathbf{a}_i) \quad = \quad \alpha_i \mathbf{a}_i + \alpha_j \mathbf{a}_j,$$

Claim. *Let $A, B \in \mathbb{F}^{n\times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). Reminder: $B \overset{R_j \to R_j + \alpha R_i}{\sim} A$. WTS $\text{Row}(A) \subseteq \text{Row}(B)$.

Now, fix $\mathbf{v} \in \text{Row}(A)$. Then $\exists \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{v} = \alpha_1 \mathbf{a}_1 + \dots + \alpha_n \mathbf{a}_n$. We now set $\beta_i := \alpha_i - \alpha_j \alpha$, and we set $\beta_k := \alpha_k$ for all $k \in \{1, \dots, n\} \setminus \{i\}$. Then

$$\beta_i \mathbf{b}_i + \beta_j \mathbf{b}_j \quad = \quad (\alpha_i - \alpha_j \alpha)\mathbf{a}_i + \alpha_j(\mathbf{a}_j + \alpha \mathbf{a}_i) \quad = \quad \alpha_i \mathbf{a}_i + \alpha_j \mathbf{a}_j,$$

whereas $\beta_k \mathbf{b}_k = \alpha_k \mathbf{a}_k$ for all $k \in \{1, \dots, n\} \setminus \{i, j\}$.

Claim. *Let $A, B \in \mathbb{F}^{n \times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). Reminder: $B \overset{R_j \to R_j + \alpha R_i}{\sim} A$. WTS $\text{Row}(A) \subseteq \text{Row}(B)$.

Now, fix $\mathbf{v} \in \text{Row}(A)$. Then $\exists \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{v} = \alpha_1 \mathbf{a}_1 + \dots + \alpha_n \mathbf{a}_n$. We now set $\beta_i := \alpha_i - \alpha_j \alpha$, and we set $\beta_k := \alpha_k$ for all $k \in \{1, \dots, n\} \setminus \{i\}$. Then

$$\beta_i \mathbf{b}_i + \beta_j \mathbf{b}_j \quad = \quad (\alpha_i - \alpha_j \alpha)\mathbf{a}_i + \alpha_j(\mathbf{a}_j + \alpha \mathbf{a}_i) \quad = \quad \alpha_i \mathbf{a}_i + \alpha_j \mathbf{a}_j,$$

whereas $\beta_k \mathbf{b}_k = \alpha_k \mathbf{a}_k$ for all $k \in \{1, \dots, n\} \setminus \{i, j\}$. Thus,

$$\beta_1 \mathbf{b}_1 + \dots + \beta_n \mathbf{b}_n \quad = \quad \alpha_1 \mathbf{a}_1 + \dots + \alpha_n \mathbf{a}_n \quad = \quad \mathbf{v},$$

Claim. *Let $A, B \in \mathbb{F}^{n \times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). Reminder: $B \overset{R_j \to R_j + \alpha R_i}{\sim} A$. WTS $\text{Row}(A) \subseteq \text{Row}(B)$.

Now, fix $\mathbf{v} \in \text{Row}(A)$. Then $\exists \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{v} = \alpha_1 \mathbf{a}_1 + \dots + \alpha_n \mathbf{a}_n$. We now set $\beta_i := \alpha_i - \alpha_j \alpha$, and we set $\beta_k := \alpha_k$ for all $k \in \{1, \dots, n\} \setminus \{i\}$. Then

$$\beta_i \mathbf{b}_i + \beta_j \mathbf{b}_j \quad = \quad (\alpha_i - \alpha_j \alpha)\mathbf{a}_i + \alpha_j(\mathbf{a}_j + \alpha \mathbf{a}_i) \quad = \quad \alpha_i \mathbf{a}_i + \alpha_j \mathbf{a}_j,$$

whereas $\beta_k \mathbf{b}_k = \alpha_k \mathbf{a}_k$ for all $k \in \{1, \dots, n\} \setminus \{i, j\}$. Thus,

$$\beta_1 \mathbf{b}_1 + \dots + \beta_n \mathbf{b}_n \quad = \quad \alpha_1 \mathbf{a}_1 + \dots + \alpha_n \mathbf{a}_n \quad = \quad \mathbf{v},$$

and it follows that $\mathbf{v} \in \text{Row}(B)$.

Claim. *Let $A, B \in \mathbb{F}^{n \times m}$ be matrices such that B is obtained from A by performing one elementary row operation. Then $Row(A) = Row(B)$.*

Proof of the Claim (outline, continued). Reminder: $B \overset{R_j \to R_j + \alpha R_i}{\sim} A$. WTS $\text{Row}(A) \subseteq \text{Row}(B)$.

Now, fix $\mathbf{v} \in \text{Row}(A)$. Then $\exists \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{v} = \alpha_1 \mathbf{a}_1 + \dots + \alpha_n \mathbf{a}_n$. We now set $\beta_i := \alpha_i - \alpha_j \alpha$, and we set $\beta_k := \alpha_k$ for all $k \in \{1, \dots, n\} \setminus \{i\}$. Then

$$\beta_i \mathbf{b}_i + \beta_j \mathbf{b}_j = (\alpha_i - \alpha_j \alpha)\mathbf{a}_i + \alpha_j(\mathbf{a}_j + \alpha \mathbf{a}_i) = \alpha_i \mathbf{a}_i + \alpha_j \mathbf{a}_j,$$

whereas $\beta_k \mathbf{b}_k = \alpha_k \mathbf{a}_k$ for all $k \in \{1, \dots, n\} \setminus \{i, j\}$. Thus,

$$\beta_1 \mathbf{b}_1 + \dots + \beta_n \mathbf{b}_n = \alpha_1 \mathbf{a}_1 + \dots + \alpha_n \mathbf{a}_n = \mathbf{v},$$

and it follows that $\mathbf{v} \in \text{Row}(B)$. Thus, $\text{Row}(A) \subseteq \text{Row}(B)$. ♦ □

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n \times m}$ have the same row space.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n \times m}$ have the same row space.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

Proof (outline).

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\mathrm{Row}(A)$. Moreover, $\dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$.

Proof (outline). Since $r := \mathrm{rank}(A)$ is equal to the number of non-zero rows of U, the first statement implies the second.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\mathrm{Row}(A)$. Moreover, $\dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$.

Proof (outline). Since $r := \mathrm{rank}(A)$ is equal to the number of non-zero rows of U, the first statement implies the second. Moreover, by Proposition 3.3.8, $\mathrm{Row}(A) = \mathrm{Row}(U)$.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\mathrm{Row}(A)$. Moreover, $\dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$.

Proof (outline). Since $r := \mathrm{rank}(A)$ is equal to the number of non-zero rows of U, the first statement implies the second. Moreover, by Proposition 3.3.8, $\mathrm{Row}(A) = \mathrm{Row}(U)$. So, it suffices to show that the non-zero rows of U form a basis of $\mathrm{Row}(U)$.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\mathrm{Row}(A)$. Moreover, $\dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$.

Proof (outline). Since $r := \mathrm{rank}(A)$ is equal to the number of non-zero rows of U, the first statement implies the second. Moreover, by Proposition 3.3.8, $\mathrm{Row}(A) = \mathrm{Row}(U)$. So, it suffices to show that the non-zero rows of U form a basis of $\mathrm{Row}(U)$.

Let $\mathbf{u}_1, \dots, \mathbf{u}_r$ be the non-zero rows of U, appearing in that order (from top to bottom) in U.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

Proof (outline). Since $r := \text{rank}(A)$ is equal to the number of non-zero rows of U, the first statement implies the second. Moreover, by Proposition 3.3.8, $\text{Row}(A) = \text{Row}(U)$. So, it suffices to show that the non-zero rows of U form a basis of $\text{Row}(U)$.

Let $\mathbf{u}_1, \dots, \mathbf{u}_r$ be the non-zero rows of U, appearing in that order (from top to bottom) in U. WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is a basis of $\text{Row}(U)$.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

Proof (outline). Since $r := \text{rank}(A)$ is equal to the number of non-zero rows of U, the first statement implies the second. Moreover, by Proposition 3.3.8, $\text{Row}(A) = \text{Row}(U)$. So, it suffices to show that the non-zero rows of U form a basis of $\text{Row}(U)$.

Let $\mathbf{u}_1, \dots, \mathbf{u}_r$ be the non-zero rows of U, appearing in that order (from top to bottom) in U. WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is a basis of $\text{Row}(U)$. Clearly, $\text{Row}(U) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_r)$.

Proposition 3.3.8

Let $\mathbb{F}$ be a field. Then any two row equivalent matrices in $\mathbb{F}^{n\times m}$ have the same row space.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

Proof (outline). Since $r := \text{rank}(A)$ is equal to the number of non-zero rows of U, the first statement implies the second. Moreover, by Proposition 3.3.8, $\text{Row}(A) = \text{Row}(U)$. So, it suffices to show that the non-zero rows of U form a basis of $\text{Row}(U)$.

Let $\mathbf{u}_1, \dots, \mathbf{u}_r$ be the non-zero rows of U, appearing in that order (from top to bottom) in U. WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is a basis of $\text{Row}(U)$. Clearly, $\text{Row}(U) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_r)$. It remains to show that $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is a linearly independent set.

Proof (outline, continued.) Reminder: U is in REF; $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are the non-zero rows of U (from top to bottom). WTS $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is linearly independent.

Proof (outline, continued.) Reminder: U is in REF; $\mathbf{u}_1, \dots, \mathbf{u}_r$ are the non-zero rows of U (from top to bottom). WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is linearly independent.

Fix scalars $\alpha_1, \dots, \alpha_r \in \mathbb{F}$ s.t. $\alpha_1 \mathbf{u}_1 + \dots + \alpha_r \mathbf{u}_r = \mathbf{0}$. WTS $\alpha_1 = \dots = \alpha_r = 0$.

Proof (outline, continued.) Reminder: U is in REF; $\mathbf{u}_1, \dots, \mathbf{u}_r$ are the non-zero rows of U (from top to bottom). WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is linearly independent.

Fix scalars $\alpha_1, \dots, \alpha_r \in \mathbb{F}$ s.t. $\alpha_1 \mathbf{u}_1 + \dots + \alpha_r \mathbf{u}_r = \mathbf{0}$. WTS $\alpha_1 = \dots = \alpha_r = 0$. Suppose otherwise, and let $i \in \{1, \dots, r\}$ be the smallest index such that $\alpha_i \neq 0$.

Proof (outline, continued.) Reminder: U is in REF; $\mathbf{u}_1, \dots, \mathbf{u}_r$ are the non-zero rows of U (from top to bottom). WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is linearly independent.

Fix scalars $\alpha_1, \dots, \alpha_r \in \mathbb{F}$ s.t. $\alpha_1\mathbf{u}_1 + \dots + \alpha_r\mathbf{u}_r = \mathbf{0}$. WTS $\alpha_1 = \dots = \alpha_r = 0$. Suppose otherwise, and let $i \in \{1, \dots, r\}$ be the smallest index such that $\alpha_i \neq 0$. So:

- $\alpha_1 = \dots = \alpha_{i-1} = 0$;
- $\alpha_i\mathbf{u}_i + \dots + \alpha_r\mathbf{u}_r = \alpha_1\mathbf{u}_1 + \dots + \alpha_r\mathbf{u}_r = \mathbf{0}$.

Proof (outline, continued.) Reminder: U is in REF; $\mathbf{u}_1, \dots, \mathbf{u}_r$ are the non-zero rows of U (from top to bottom). WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is linearly independent.

Fix scalars $\alpha_1, \dots, \alpha_r \in \mathbb{F}$ s.t. $\alpha_1 \mathbf{u}_1 + \dots + \alpha_r \mathbf{u}_r = \mathbf{0}$. WTS $\alpha_1 = \dots = \alpha_r = 0$. Suppose otherwise, and let $i \in \{1, \dots, r\}$ be the smallest index such that $\alpha_i \neq 0$. So:

- $\alpha_1 = \dots = \alpha_{i-1} = 0$;
- $\alpha_i \mathbf{u}_i + \dots + \alpha_r \mathbf{u}_r = \alpha_1 \mathbf{u}_1 + \dots + \alpha_r \mathbf{u}_r = \mathbf{0}$.

WMA that the leading entry (i.e. the leftmost non-zero entry) of the row $\mathbf{u}_i$ is in position j.

$$\begin{array}{cc}
 & \begin{array}{cccccccccccc} & & & & & & & j & & & & \\ & & & & & & & \downarrow & & & & \end{array} \\
 \begin{array}{c} \\ \\ \\ i \rightarrow \\ \\ \\ \\ \\ \end{array} &
\begin{bmatrix}
0 & \blacksquare & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

Proof (outline, continued.) Reminder: U is in REF; $\mathbf{u}_1, \dots, \mathbf{u}_r$ are the non-zero rows of U (from top to bottom). WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is linearly independent.

Fix scalars $\alpha_1, \dots, \alpha_r \in \mathbb{F}$ s.t. $\alpha_1 \mathbf{u}_1 + \dots + \alpha_r \mathbf{u}_r = \mathbf{0}$. WTS $\alpha_1 = \dots = \alpha_r = 0$. Suppose otherwise, and let $i \in \{1, \dots, r\}$ be the smallest index such that $\alpha_i \neq 0$. So:

- $\alpha_1 = \dots = \alpha_{i-1} = 0$;
- $\alpha_i \mathbf{u}_i + \dots + \alpha_r \mathbf{u}_r = \alpha_1 \mathbf{u}_1 + \dots + \alpha_r \mathbf{u}_r = \mathbf{0}$.

WMA that the leading entry (i.e. the leftmost non-zero entry) of the row $\mathbf{u}_i$ is in position j.

$$\begin{array}{cc}
 & \begin{matrix} & & & & & & & j & & & & \\ & & & & & & & \downarrow & & & & \end{matrix} \\
 \begin{matrix} \\ \\ \\ i \rightarrow \\ \\ \\ \\ \\ \end{matrix} &
\begin{bmatrix}
0 & \blacksquare & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

But then the j-th entry of $\alpha_i \mathbf{u}_i + \dots + \alpha_r \mathbf{u}_r$ is non-zero, contrary to the fact that $\alpha_i \mathbf{u}_i + \dots + \alpha_r \mathbf{u}_r = \mathbf{0}$. $\square$

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\text{Col}(A)$. Moreover, $\dim(\text{Col}(A)) = \text{rank}(A)$.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then the pivot columns of A form a basis of $\text{Col}(A)$. Moreover, $\dim(\text{Col}(A)) = \text{rank}(A)$.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

Corollary 3.3.11

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then both the following hold:

(a) $\dim(\text{Col}(A)) = \dim(\text{Row}(A)) = \text{rank}(A)$;

(b) $\text{rank}(A) = \text{rank}(A^T)$.

Proof.

Theorem 3.3.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the pivot columns of A form a basis of $\text{Col}(A)$. Moreover, $\dim(\text{Col}(A)) = \text{rank}(A)$.

Theorem 3.3.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$, and let U be any matrix in row echelon form that is row equivalent to A. Then the non-zero rows of U form a basis of $\text{Row}(A)$. Moreover, $\dim(\text{Row}(A)) = \text{rank}(A)$.

Corollary 3.3.11

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then both the following hold:

(a) $\dim(\text{Col}(A)) = \dim(\text{Row}(A)) = \text{rank}(A)$;

(b) $\text{rank}(A) = \text{rank}(A^T)$.

Proof. Part (a) follows immediately from Theorems 3.3.4 and 3.3.9.

Corollary 3.3.11

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then both the following hold:

(a) $\dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$;

(b) $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Proof (continued). For (b), we observe that

$$\mathrm{rank}(A) \overset{(a)}{=} \dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A^T)) \overset{(a)}{=} \mathrm{rank}(A^T),$$

and we are done. $\square$

2. Matrices of full rank

2. Matrices of full rank

- We simply state a theorem and a corollary (without proof).
  - The theorem and the corollary essentially summarize various results that we have obtained so far.
  - The full proof is in the Lecture Notes (and essentially consists of references to previously proven results).

Theorem 3.3.14

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) the columns of A are linearly independent iff $\text{rank}(A) = m$ (i.e. A has full column rank);

(b) the columns of A span $\mathbb{F}^n$ (i.e. $\text{Col}(A) = \mathbb{F}^n$) iff $\text{rank}(A) = n$ (i.e. A has full row rank);

(c) the rows of A are linearly independent iff $\text{rank}(A) = n$ (i.e. A has full row rank);

(d) the rows of A span $\mathbb{F}^{1 \times m}$ (i.e. $\text{Row}(A) = \mathbb{F}^{1 \times m}$) iff $\text{rank}(A) = m$ (i.e. A has full column rank).

- Remarks:
  - Parts (a) and (b) were proven in an earlier lecture.
  - To obtain (c) and (d), we apply (a) and (b), respectively, to the matrix A^T, and we use the fact that $\text{rank}(A^T) = \text{rank}(A)$.

Corollary 3.3.15

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be a **square** matrix. Then the following are equivalent:

(a) $\text{rank}(A) = n$;

(b) $\text{rank}(A^T) = n$;

(c) the columns of A are linearly independent;

(d) the columns of A span $\mathbb{F}^n$ (i.e. $\text{Col}(A) = \mathbb{F}^n$);

(e) the columns of A form a basis of $\mathbb{F}^n$;

(f) the rows of A are linearly independent;

(g) the rows of A span $\mathbb{F}^{1 \times n}$ (i.e. $\text{Row}(A) = \mathbb{F}^{1 \times n}$);

(h) the rows of A form a basis of $\mathbb{F}^{1 \times n}$.

3. The rank of matrix products. Left and right inverses of a matrix

3. The rank of matrix products. Left and right inverses of a matrix

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof.

3. The rank of matrix products. Left and right inverses of a matrix

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof. We first prove (a).

3. The rank of matrix products. Left and right inverses of a matrix

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof. We first prove (a). Fix an invertible matrix $S \in \mathbb{F}^{n \times n}$.

3. The rank of matrix products. Left and right inverses of a matrix

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof. We first prove (a). Fix an invertible matrix $S \in \mathbb{F}^{n \times n}$. By Theorem 1.11.13, A and SA are row equivalent,

3. The rank of matrix products. Left and right inverses of a matrix

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

a. for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;
b. for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;
c. for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof. We first prove (a). Fix an invertible matrix $S \in \mathbb{F}^{n \times n}$. By Theorem 1.11.13, A and SA are row equivalent, and so by Proposition 1.6.2, they have the same rank.

3. The rank of matrix products. Left and right inverses of a matrix

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof. We first prove (a). Fix an invertible matrix $S \in \mathbb{F}^{n \times n}$. By Theorem 1.11.13, A and SA are row equivalent, and so by Proposition 1.6.2, they have the same rank. This proves (a).

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof (continued). We now prove (b).

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof (continued). We now prove (b). Fix an invertible matrix $S \in \mathbb{F}^{m \times m}$.

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n\times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m\times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n\times n}$ and $S_2 \in \mathbb{F}^{m\times m}$, we have that $\text{rank}(S_1AS_2) = \text{rank}(A)$.

Proof (continued). We now prove (b). Fix an invertible matrix $S \in \mathbb{F}^{m\times m}$. Then by the Invertible Matrix Theorem (version 1), S^T is also invertible.

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof (continued). We now prove (b). Fix an invertible matrix $S \in \mathbb{F}^{m \times m}$. Then by the Invertible Matrix Theorem (version 1), S^T is also invertible. We now compute:

$$\begin{array}{rcll} \text{rank}(AS) & = & \text{rank}\big((AS)^T\big) & \text{by Corollary 3.3.11(b)} \\ & = & \text{rank}\big(S^T A^T\big) & \text{by Proposition 1.8.1(d)} \\ & = & \text{rank}\big(A^T\big) & \text{by (a), since } S^T \text{ is invertible} \\ & = & \text{rank}(A) & \text{by Corollary 3.3.11(b).} \end{array}$$

This proves (b).

Proposition 3.3.16

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then all the following hold:

(a) for all invertible matrices $S \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(SA) = \text{rank}(A)$;

(b) for all invertible matrices $S \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(AS) = \text{rank}(A)$;

(c) for all invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, we have that $\text{rank}(S_1 A S_2) = \text{rank}(A)$.

Proof (continued). Finally, for (c), we fix invertible matrices $S_1 \in \mathbb{F}^{n \times n}$ and $S_2 \in \mathbb{F}^{m \times m}$, and we observe that

$$\text{rank}(S_1 A S_2) \stackrel{(a)}{=} \text{rank}(A S_2) \stackrel{(b)}{=} \text{rank}(A),$$

and we are done. □

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof.

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_p \end{bmatrix}$.

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof. Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$ and $B = \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_p \end{array} \right]$. WTS $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_p \end{bmatrix}$. WTS $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

We first prove that $\text{rank}(AB) \leq \text{rank}(A)$.

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof. Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array}\right]$ and $B = \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_p \end{array}\right]$. WTS $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

We first prove that $\text{rank}(AB) \leq \text{rank}(A)$. By definition, we have that $AB = \left[\begin{array}{ccc} A\mathbf{b}_1 & \dots & A\mathbf{b}_p \end{array}\right]$,

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof. Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array}\right]$ and $B = \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_p \end{array}\right]$. WTS $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

We first prove that $\text{rank}(AB) \leq \text{rank}(A)$. By definition, we have that $AB = \left[\begin{array}{ccc} A\mathbf{b}_1 & \dots & A\mathbf{b}_p \end{array}\right]$, and in particular, every column of AB is a linear combination of the columns of A,

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof. Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array}\right]$ and $B = \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_p \end{array}\right]$. WTS $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

We first prove that $\text{rank}(AB) \leq \text{rank}(A)$. By definition, we have that $AB = \left[\begin{array}{ccc} A\mathbf{b}_1 & \dots & A\mathbf{b}_p \end{array}\right]$, and in particular, every column of AB is a linear combination of the columns of A, i.e. every column of AB belongs to $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \text{Col}(A)$.

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof. Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array}\right]$ and $B = \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_p \end{array}\right]$. WTS $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

We first prove that $\text{rank}(AB) \leq \text{rank}(A)$. By definition, we have that $AB = \left[\begin{array}{ccc} A\mathbf{b}_1 & \dots & A\mathbf{b}_p \end{array}\right]$, and in particular, every column of AB is a linear combination of the columns of A, i.e. every column of AB belongs to $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \text{Col}(A)$.

Since $\text{Col}(A)$ is a subspace of $\mathbb{F}^n$ (and is therefore a vector space in its own right), Theorem 3.1.11(b) now guarantees that $\text{Col}(AB) = \text{Span}(A\mathbf{b}_1, \dots, A\mathbf{b}_p)$ is a subspace of $\text{Col}(A)$.

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_p \end{bmatrix}$. WTS $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

We first prove that $\text{rank}(AB) \leq \text{rank}(A)$. By definition, we have that $AB = \begin{bmatrix} A\mathbf{b}_1 & \dots & A\mathbf{b}_p \end{bmatrix}$, and in particular, every column of AB is a linear combination of the columns of A, i.e. every column of AB belongs to $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \text{Col}(A)$.

Since $\text{Col}(A)$ is a subspace of $\mathbb{F}^n$ (and is therefore a vector space in its own right), Theorem 3.1.11(b) now guarantees that $\text{Col}(AB) = \text{Span}(A\mathbf{b}_1, \dots, A\mathbf{b}_p)$ is a subspace of $\text{Col}(A)$. Since $\text{Col}(A)$ is finite-dimensional, Theorem 3.2.21 now implies that

$$\dim\big(\text{Col}(AB)\big) \quad \leq \quad \dim\big(\text{Col}(A)\big),$$

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof. Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$ and $B = \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_p \end{array} \right]$. WTS $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

We first prove that $\text{rank}(AB) \leq \text{rank}(A)$. By definition, we have that $AB = \left[\begin{array}{ccc} A\mathbf{b}_1 & \dots & A\mathbf{b}_p \end{array} \right]$, and in particular, every column of AB is a linear combination of the columns of A, i.e. every column of AB belongs to $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \text{Col}(A)$.

Since $\text{Col}(A)$ is a subspace of $\mathbb{F}^n$ (and is therefore a vector space in its own right), Theorem 3.1.11(b) now guarantees that $\text{Col}(AB) = \text{Span}(A\mathbf{b}_1, \dots, A\mathbf{b}_p)$ is a subspace of $\text{Col}(A)$. Since $\text{Col}(A)$ is finite-dimensional, Theorem 3.2.21 now implies that

$$\text{dim}\big(\text{Col}(AB)\big) \quad \leq \quad \text{dim}\big(\text{Col}(A)\big),$$

and we deduce that (next slide):

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof (continued).

$$\text{rank}(AB) \quad \stackrel{(*)}{=} \quad \dim\big(\text{Col}(AB)\big) \quad \leq \quad \dim\big(\text{Col}(A)\big) \quad \stackrel{(*)}{=} \quad \text{rank}(A),$$

where both instances of (*) follow from Theorem 3.3.4 (or alternatively, from Corollary 3.3.11(a)).

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof (continued). We have now shown that $\text{rank}(AB) \leq \text{rank}(A)$.

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof (continued). We have now shown that $\text{rank}(AB) \leq \text{rank}(A)$. A completely analogous argument shows that
$\text{rank}(B^T A^T) \leq \text{rank}(B^T)$,

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times p}$. Then

$$\text{rank}(AB) \quad \leq \quad \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}.$$

Proof (continued). We have now shown that $\text{rank}(AB) \leq \text{rank}(A)$. A completely analogous argument shows that $\text{rank}(B^T A^T) \leq \text{rank}(B^T)$, and we deduce that

$$\begin{array}{rcl} \text{rank}(AB) & \stackrel{(*)}{=} & \text{rank}((AB)^T) \\ & = & \text{rank}(B^T A^T) \\ & \leq & \text{rank}(B^T) \\ & \stackrel{(*)}{=} & \text{rank}(B) \end{array}$$

where both instances of (*) follow from Corollary 3.3.11(b). □

Corollary 3.3.18

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then AB is invertible iff A and B are both invertible.

Proof.

Corollary 3.3.18

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then AB is invertible iff A and B are both invertible.

Proof. If A and B are invertible, then Proposition 1.11.8(d) guarantees that AB is invertible.

Corollary 3.3.18

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then AB is invertible iff A and B are both invertible.

Proof. If A and B are invertible, then Proposition 1.11.8(d) guarantees that AB is invertible.

For the other direction, assume that AB is invertible. Then by the Invertible Matrix Theorem (version 1), we have that $\text{rank}(AB) = n$, and it suffices to show that $\text{rank}(A) = n$ and $\text{rank}(B) = n$.

Corollary 3.3.18

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Then AB is invertible iff A and B are both invertible.

Proof. If A and B are invertible, then Proposition 1.11.8(d) guarantees that AB is invertible.

For the other direction, assume that AB is invertible. Then by the Invertible Matrix Theorem (version 1), we have that $\text{rank}(AB) = n$, and it suffices to show that $\text{rank}(A) = n$ and $\text{rank}(B) = n$.

By Theorem 3.3.17, we have that
$n = \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$,

Corollary 3.3.18

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then AB is invertible iff A and B are both invertible.

Proof. If A and B are invertible, then Proposition 1.11.8(d) guarantees that AB is invertible.

For the other direction, assume that AB is invertible. Then by the Invertible Matrix Theorem (version 1), we have that $\mathrm{rank}(AB) = n$, and it suffices to show that $\mathrm{rank}(A) = n$ and $\mathrm{rank}(B) = n$.

By Theorem 3.3.17, we have that $n = \mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$, and it follows that $\mathrm{rank}(A) \geq n$ and $\mathrm{rank}(B) \geq n$.

Corollary 3.3.18

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Then AB is invertible iff A and B are both invertible.

Proof. If A and B are invertible, then Proposition 1.11.8(d) guarantees that AB is invertible.

For the other direction, assume that AB is invertible. Then by the Invertible Matrix Theorem (version 1), we have that $\text{rank}(AB) = n$, and it suffices to show that $\text{rank}(A) = n$ and $\text{rank}(B) = n$.

By Theorem 3.3.17, we have that $n = \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$, and it follows that $\text{rank}(A) \geq n$ and $\text{rank}(B) \geq n$. But since A and B are $n \times n$ matrices, Proposition 1.6.3 now implies that $\text{rank}(A) = n$ and $\text{rank}(B) = n$, and we are done. $\square$

Definition

Suppose that $A \in \mathbb{F}^{n \times m}$, where $\mathbb{F}$ is some field. A *left inverse* of A is a matrix $B \in \mathbb{F}^{m \times n}$ such that $BA = I_m$, and a *right inverse* of A is a matrix $C \in \mathbb{F}^{m \times n}$ such that $AC = I_n$.

- Thus, a left inverse (resp. right inverse) of a matrix A is a matrix that we can multiply A by on the left (resp. on the right) in order to obtain the identity matrix of the appropriate size.

Definition

Suppose that $A \in \mathbb{F}^{n \times m}$, where $\mathbb{F}$ is some field. A *left inverse* of A is a matrix $B \in \mathbb{F}^{m \times n}$ such that $BA = I_m$, and a *right inverse* of A is a matrix $C \in \mathbb{F}^{m \times n}$ such that $AC = I_n$.

- Thus, a left inverse (resp. right inverse) of a matrix A is a matrix that we can multiply A by on the left (resp. on the right) in order to obtain the identity matrix of the appropriate size.
- Consider, for example, matrices

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Then $A_1 A_2 = I_2$, and consequently, A_1 is a left inverse of A_2, and A_2 is a right inverse of A_1.

Definition

Suppose that $A \in \mathbb{F}^{n \times m}$, where $\mathbb{F}$ is some field. A *left inverse* of A is a matrix $B \in \mathbb{F}^{m \times n}$ such that $BA = I_m$, and a *right inverse* of A is a matrix $C \in \mathbb{F}^{m \times n}$ such that $AC = I_n$.

- Obviously, a matrix need not have a left or a right inverse.
  - For example, zero matrices have no left inverses and no right inverses.

Definition

Suppose that $A \in \mathbb{F}^{n\times m}$, where $\mathbb{F}$ is some field. A *left inverse* of A is a matrix $B \in \mathbb{F}^{m\times n}$ such that $BA = I_m$, and a *right inverse* of A is a matrix $C \in \mathbb{F}^{m\times n}$ such that $AC = I_n$.

- Obviously, a matrix need not have a left or a right inverse.
  - For example, zero matrices have no left inverses and no right inverses.
- On the other hand, a matrix may possibly have more than one left inverse or more than one right inverse.

Definition

Suppose that $A \in \mathbb{F}^{n \times m}$, where $\mathbb{F}$ is some field. A *left inverse* of A is a matrix $B \in \mathbb{F}^{m \times n}$ such that $BA = I_m$, and a *right inverse* of A is a matrix $C \in \mathbb{F}^{m \times n}$ such that $AC = I_n$.

- Obviously, a matrix need not have a left or a right inverse.
  - For example, zero matrices have no left inverses and no right inverses.
- On the other hand, a matrix may possibly have more than one left inverse or more than one right inverse.
- However, as Corollary 3.3.19 (next slide) shows, any matrix A that has both a left inverse and a right inverse is in fact invertible (and in particular, square), and moreover, both its left inverse and its right inverse are unique are equal to A^{-1}.

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m \times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m \times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

Proof.

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m \times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m \times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

Proof. First, we have that

$$m = \operatorname{rank}(I_m) \stackrel{(*)}{=} \operatorname{rank}(BA) \stackrel{(**)}{\leq} \min\{\operatorname{rank}(B), \operatorname{rank}(A)\} \stackrel{(***)}{\leq} n,$$

where (*) follows from the fact that $BA = I_m$, (**) follows from Theorem 3.3.17, and (***) follows from Proposition 1.6.3 (because A is an $n \times m$ matrix and B is an $m \times n$ matrix).

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m\times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m\times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

Proof. First, we have that

$$m \;=\; \mathrm{rank}(I_m) \;\overset{(*)}{=}\; \mathrm{rank}(BA) \;\overset{(**)}{\leq}\; \min\left\{\mathrm{rank}(B), \mathrm{rank}(A)\right\} \;\overset{(***)}{\leq}\; n,$$

where (*) follows from the fact that $BA = I_m$, (**) follows from Theorem 3.3.17, and (***) follows from Proposition 1.6.3 (because A is an $n \times m$ matrix and B is an $m \times n$ matrix).

Since $AC = I_n$, an analogous argument establishes that $n \leq m$ (we simply use the fact that $AC = I_n$ instead of $BA = I_m$).

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m\times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m\times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

Proof. First, we have that

$$m \;=\; \mathrm{rank}(I_m) \;\stackrel{(*)}{=}\; \mathrm{rank}(BA) \;\stackrel{(**)}{\leq}\; \min\left\{\mathrm{rank}(B), \mathrm{rank}(A)\right\} \;\stackrel{(***)}{\leq}\; n,$$

where (*) follows from the fact that $BA = I_m$, (**) follows from Theorem 3.3.17, and (***) follows from Proposition 1.6.3 (because A is an $n \times m$ matrix and B is an $m \times n$ matrix).

Since $AC = I_n$, an analogous argument establishes that $n \leq m$ (we simply use the fact that $AC = I_n$ instead of $BA = I_m$).

So, $m = n$.

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m \times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m \times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

Proof. First, we have that

$$m \quad = \quad \text{rank}(I_m) \quad \overset{(*)}{=} \quad \text{rank}(BA) \quad \overset{(**)}{\leq} \quad \min\left\{\text{rank}(B), \text{rank}(A)\right\} \quad \overset{(***)}{\leq} \quad n,$$

where (*) follows from the fact that $BA = I_m$, (**) follows from Theorem 3.3.17, and (***) follows from Proposition 1.6.3 (because A is an $n \times m$ matrix and B is an $m \times n$ matrix).

Since $AC = I_n$, an analogous argument establishes that $n \leq m$ (we simply use the fact that $AC = I_n$ instead of $BA = I_m$).

So, $m = n$. In particular, we have that $A, B, C \in \mathbb{F}^{n \times n}$, and that $BA = I_m = I_n$ and $AC = I_n$.

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m \times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m \times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

Proof. Reminder: $n = m$; $BA = AC = I_n$.

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m\times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m\times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

Proof. Reminder: $n = m$; $BA = AC = I_n$.

But now

$$B \;=\; BI_n \;=\; B(\underbrace{AC}_{=I_n}) \;=\; (\underbrace{BA}_{=I_n})C \;=\; I_nC \;=\; C.$$

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m \times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m \times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

Proof. Reminder: $n = m$; $BA = AC = I_n$.

But now

$$B \;=\; BI_n \;=\; B(\underbrace{AC}_{=I_n}) \;=\; (\underbrace{BA}_{=I_n})C \;=\; I_nC \;=\; C.$$

So, $AB = BA = I_n$.

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m \times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m \times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

Proof. Reminder: $n = m$; $BA = AC = I_n$.

But now

$$B \quad = \quad BI_n \quad = \quad B(\underbrace{AC}_{=I_n}) \quad = \quad (\underbrace{BA}_{=I_n})C \quad = \quad I_nC \quad = \quad C.$$

So, $AB = BA = I_n$. Thus, A is invertible, and its inverse is $B = C$. This completes the argument. $\square$

Corollary 3.3.19

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times m}$ be a matrix, and assume that $B \in \mathbb{F}^{m \times n}$ is a left inverse of A (i.e. $BA = I_m$) and that $C \in \mathbb{F}^{m \times n}$ is a right inverse of A (i.e. $AC = I_n$). Then A is invertible (and in particular square, i.e. $m = n$), and $B = C = A^{-1}$.

- Remark: Corollary 3.3.19 is the reason that we defined invertibility only for square matrices.
  - Any reasonable definition of an invertible matrix would entail the existence of both a left and a right inverse for that matrix, and by Corollary 3.3.19, only square matrices can have both a left and a right inverse.

- Reminder:

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$. Then

$$\operatorname{rank}(AB) \quad \leq \quad \min\Big\{\operatorname{rank}(A), \operatorname{rank}(B)\Big\}.$$

- Reminder:

Theorem 3.3.17

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times p}$. Then
$$\operatorname{rank}(AB) \quad \leq \quad \min\Big\{\operatorname{rank}(A), \operatorname{rank}(B)\Big\}.$$

- As a corollary of Theorem 3.3.17 for **square** matrices, we get the following.

Corollary 3.3.20

Let $\mathbb{F}$ be field, and let $A, B \in \mathbb{F}^{n\times n}$ be such that $AB = I_n$ or $BA = I_n$. Then $AB = BA = I_n$, i.e. A and B are both invertible and are each other's inverses.

- Proof: Lecture Notes.
- Remark: Note that Corollary 3.3.20 implies that if a square matrix A has a left **or** a right inverse B, then B is in fact a "two-sided inverse" of A, i.e. the (ordinary) inverse of A, and in particular, A is invertible.

4. The null space of a matrix

4. The null space of a matrix

Definition

For field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n \times m}$, we define the *null space* of A, denoted by $\mathrm{Nul}(A)$, to be the set of all solutions of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$, i.e.

$$\mathrm{Nul}(A) \;:=\; \{\mathbf{x} \in \mathbb{F}^m \mid A\mathbf{x} = \mathbf{0}\}.$$

- Notation: In some texts, notation $\mathrm{Ker}(A)$ is used instead of $\mathrm{Nul}(A)$. "Ker" stands for "kernel."

4. The null space of a matrix

Definition

For field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n\times m}$, we define the *null space* of A, denoted by $\mathrm{Nul}(A)$, to be the set of all solutions of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$, i.e.

$$\mathrm{Nul}(A) \quad := \quad \{\mathbf{x} \in \mathbb{F}^m \mid A\mathbf{x} = \mathbf{0}\}.$$

- Notation: In some texts, notation $\mathrm{Ker}(A)$ is used instead of $\mathrm{Nul}(A)$. "Ker" stands for "kernel."

Proposition 3.3.25

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then $\mathrm{Nul}(A)$ is a subspace of $\mathbb{F}^m$.

- Proof: Lecture Notes.

4. The null space of a matrix

Definition

For field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n \times m}$, we define the *null space* of A, denoted by $\mathrm{Nul}(A)$, to be the set of all solutions of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$, i.e.

$$\mathrm{Nul}(A) \quad := \quad \{\mathbf{x} \in \mathbb{F}^m \mid A\mathbf{x} = \mathbf{0}\}.$$

- Notation: In some texts, notation $\mathrm{Ker}(A)$ is used instead of $\mathrm{Nul}(A)$. "Ker" stands for "kernel."

Proposition 3.3.25

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then $\mathrm{Nul}(A)$ is a subspace of $\mathbb{F}^m$.

- Proof: Lecture Notes.
- Terminology: The dimension of $\mathrm{Nul}(A)$ is called the *nullity* of the matrix A.

Proposition 3.3.26

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{m \times n}$. Then the columns of A are linearly independent iff $\mathrm{Nul}(A) = \{\mathbf{0}\}$.

Proof.

Proposition 3.3.26

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{m \times n}$. Then the columns of A are linearly independent iff $\mathrm{Nul}(A) = \{\mathbf{0}\}$.

Proof. By definition, $\mathrm{Nul}(A)$ is the set of all solutions of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$; consequently,

$$\mathrm{Nul}(A) = \{\mathbf{0}\} \iff \text{the homogeneous matrix-vector equation } A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution (i.e. the sol'n } \mathbf{x} = \mathbf{0})$$

$$\overset{(*)}{\iff} \text{the columns of } A \text{ are linearly independent,}$$

where (*) follows from Proposition 3.2.1. □

Example 3.3.27

Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_2$. Find a basis of $\mathrm{Nul}(A)$. What is $\dim(\mathrm{Nul}(A))$?

Solution.

Example 3.3.27

Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_2$. Find a basis of $\mathrm{Nul}(A)$. What is $\dim(\mathrm{Nul}(A))$?

Solution. We begin by finding the general solution of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$.

Example 3.3.27

Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_2$. Find a basis of $\mathrm{Nul}(A)$. What is $\dim(\mathrm{Nul}(A))$?

Solution. We begin by finding the general solution of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$. By row reducing, we get

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution (continued). Reminder:

$$\mathrm{RREF}(A) \quad = \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution (continued). Reminder:

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} r+t \\ s \\ \boxed{\text{r}} \\ \boxed{\text{s}} \\ \boxed{\text{t}} \end{bmatrix}, \qquad \text{where } r, s, t \in \mathbb{Z}_2,$$

Solution (continued). Reminder:

$$\mathrm{RREF}(A) \;=\; \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} \;=\; \begin{bmatrix} r+t \\ s \\ \boxed{\mathrm{r}} \\ \boxed{\mathrm{s}} \\ \boxed{\mathrm{t}} \end{bmatrix}, \qquad \text{where } r, s, t \in \mathbb{Z}_2,$$

that is,

$$\mathbf{x} \;=\; r \begin{bmatrix} 1 \\ 0 \\ \boxed{1} \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ \boxed{1} \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \boxed{1} \end{bmatrix}, \qquad \text{where } r, s, t \in \mathbb{Z}_2.$$

Solution (continued). Reminder:

$$\mathbf{x} = r\begin{bmatrix}1\\0\\\boxed{1}\\0\\0\end{bmatrix} + s\begin{bmatrix}0\\1\\0\\\boxed{1}\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\0\\0\\\boxed{1}\end{bmatrix}, \qquad \text{where } r, s, t \in \mathbb{Z}_2.$$

Solution (continued). Reminder:

$$\mathbf{x} \;=\; r\begin{bmatrix} 1 \\ 0 \\ \boxed{1} \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \\ \boxed{1} \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \boxed{1} \end{bmatrix}, \qquad \text{where } r, s, t \in \mathbb{Z}_2.$$

So,

$$\mathcal{B} \;:=\; \left\{ \begin{bmatrix} 1 \\ 0 \\ \boxed{1} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \boxed{1} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \boxed{1} \end{bmatrix} \right\}$$

is a basis of $\mathrm{Nul}(A)$, and it follows that $\dim\big(\mathrm{Nul}(A)\big) = 3$. □

- Remark: Let $A \in \mathbb{F}^{n \times m}$ (where $\mathbb{F}$ is some field).

- Remark: Let $A \in \mathbb{F}^{n\times m}$ (where $\mathbb{F}$ is some field).
  1. We note that $\dim\big(\mathrm{Nul}(A)\big)$ (i.e. the number of vectors in any basis of $\mathrm{Nul}(A)$) will always be equal to the number of free variables in the general solution of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$.
     - We omit a fully formal proof of this, but the basic argument is as in our solution of Example 3.3.27.

- Remark: Let $A \in \mathbb{F}^{n\times m}$ (where $\mathbb{F}$ is some field).
  1. We note that $\dim\big(\mathrm{Nul}(A)\big)$ (i.e. the number of vectors in any basis of $\mathrm{Nul}(A)$) will always be equal to the number of free variables in the general solution of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$.
     - We omit a fully formal proof of this, but the basic argument is as in our solution of Example 3.3.27.
  2. If the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, i.e. the solution $\mathbf{x} = \mathbf{0}$, then $\mathrm{Nul}(A) = \{\mathbf{0}\}$, and $\emptyset$ is the (unique) basis of $\mathrm{Nul}(A)$.

The rank–nullity theorem (matrix version).

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$. Then

$$\operatorname{rank}(A) + \dim\big(\operatorname{Nul}(A)\big) = \underbrace{m}_{\substack{=\text{ number of}\\ \text{columns of } A}}.$$

- An informal diagram summarizing the rank-nullity theorem (matrix version):

$$\underbrace{\operatorname{rank}(A)}_{\substack{=\text{ number of}\\ \text{pivot}\\ \text{columns of } A\\ \\ =\text{ number of}\\ \text{basic variables}}} + \underbrace{\dim\big(\operatorname{Nul}(A)\big)}_{\substack{=\text{ number of}\\ \text{non-pivot}\\ \text{columns of } A\\ \\ =\text{ number of}\\ \text{free variables}}} = \underbrace{m}_{\substack{=\text{ number of}\\ \text{columns of } A}}$$

5. The Invertible Matrix Theorem (version 2)

5. The Invertible Matrix Theorem (version 2)

The Invertible Matrix Theorem (version 2)

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be a **square** matrix. Further, let $f : \mathbb{F}^n \to \mathbb{F}^n$ be given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$.[a] Then the following are equivalent:

(a) A is invertible (i.e. A has an inverse);

(b) A^T is invertible;

(c) $\text{RREF}(A) = I_n$;

(d) $\text{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right]) = \left[\begin{array}{c:c} I_n & B \end{array}\right]$ for some matrix $B \in \mathbb{F}^{n \times n}$;

(e) $\text{rank}(A) = n$;

(f) $\text{rank}(A^T) = n$;

(g) A is a product of elementary matrices;

---

[a] Since f is a matrix transformation, Proposition 1.10.4 guarantees that f is linear. Moreover, A is the standard matrix of f.

The Invertible Matrix Theorem (version 2) - continued

- (h) the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (i.e. the solution $\mathbf{x} = \mathbf{0}$);
- (i) there exists some vector $\mathbf{b} \in \mathbb{F}^n$ such that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;
- (j) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;
- (k) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has at most one solution;
- (l) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent;
- (m) f is one-to-one;
- (n) f is onto;
- (o) f is an isomorphism;

The Invertible Matrix Theorem (version 2) - continued

(p) there exists a matrix $B \in \mathbb{F}^{n\times n}$ such that $BA = I_n$ (i.e. A has a left inverse);

(q) there exists a matrix $C \in \mathbb{F}^{n\times n}$ such that $AC = I_n$ (i.e. A has a right inverse);

(r) the columns of A are linearly independent;

(s) the columns of A span $\mathbb{F}^n$ (i.e. $\mathrm{Col}(A) = \mathbb{F}^n$);

(t) the columns of A form a basis of $\mathbb{F}^n$;

(u) the rows of A are linearly independent;

(v) the rows of A span $\mathbb{F}^{1\times n}$ (i.e. $\mathrm{Row}(A) = \mathbb{F}^{1\times n}$);

(w) the rows of A form a basis of $\mathbb{F}^{1\times n}$;

(x) $\mathrm{Nul}(A) = \{\mathbf{0}\}$ (i.e. $\dim(\mathrm{Nul}(A)) = 0$).