Linear Algebra 1

Lecture #8

Vector spaces (part II)

Irena Penev

December 4, 2024

- In this lecture, we examine the relationship between the sizes of linearly independent sets, spanning sets, and bases of a finite-dimensional vector space.

- In this lecture, we examine the relationship between the sizes of linearly independent sets, spanning sets, and bases of a finite-dimensional vector space.
- In particular, we will prove the following two theorems.

- In this lecture, we examine the relationship between the sizes of linearly independent sets, spanning sets, and bases of a finite-dimensional vector space.
- In particular, we will prove the following two theorems.

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

- In this lecture, we examine the relationship between the sizes of linearly independent sets, spanning sets, and bases of a finite-dimensional vector space.
- In particular, we will prove the following two theorems.

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Definition

The *dimension* of a finite-dimensional vector space V over a field $\mathbb{F}$, denoted by $\dim(V)$, is the number of elements in any basis of V (by Theorem 3.2.16, this is well-defined).

- For this definition to make sense, we will first need to prove Theorem 3.2.16!

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

- Informally, Theorem 3.2.17 says:

$$|\text{linearly independent set of } V| \quad \leq \quad \dim(V) \quad \leq \quad |\text{spanning set of } V|.$$

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

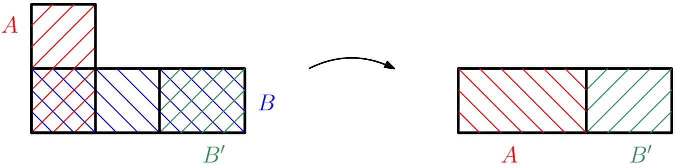
- Informally, Theorem 3.2.17 says:

$$|\text{linearly independent set of } V| \quad \leq \quad \dim(V) \quad \leq \quad |\text{spanning set of } V|.$$

- A key ingredient in the proofs of Theorems 3.2.16 and 3.2.17 is the so-called "Steinitz exchange lemma," to which we now turn.

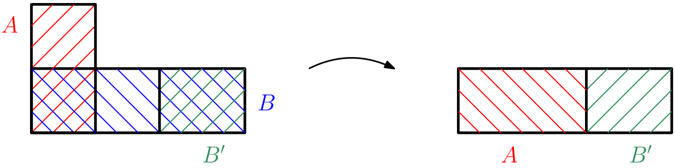
The Steinitz exchange lemma

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and assume that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then $k \leq \ell$ (i.e. $|A| \leq |B|$). Moreover, there exists a set $B' \subseteq B \setminus A$ s.t. $|B'| = |B| - |A| = \ell - k$ and $A \cup B'$ is a spanning set of V.



The Steinitz exchange lemma

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and assume that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then $k \leq \ell$ (i.e. $|A| \leq |B|$). Moreover, there exists a set $B' \subseteq B \setminus A$ s.t. $|B'| = |B| - |A| = \ell - k$ and $A \cup B'$ is a spanning set of V.



- First, some remarks. Then, a proof.

- For technical reasons (in order to get the set B'), the Steinitz exchange lemma assumes that the sets A and B contain no repetitions.

- For technical reasons (in order to get the set B'), the Steinitz exchange lemma assumes that the sets A and B contain no repetitions.
  - Actually, it would be possible to state and prove a version of the Steinitz exchange lemma that allows repetitions.

- For technical reasons (in order to get the set B'), the Steinitz exchange lemma assumes that the sets A and B contain no repetitions.
    - Actually, it would be possible to state and prove a version of the Steinitz exchange lemma that allows repetitions.
    - However, this would be notationally messy.

- For technical reasons (in order to get the set B'), the Steinitz exchange lemma assumes that the sets A and B contain no repetitions.
  - Actually, it would be possible to state and prove a version of the Steinitz exchange lemma that allows repetitions.
  - However, this would be notationally messy.
- However, if we only care about the "$|A| \leq |B|$" part of the Steinitz exchange lemma (which is what we usually care about), then this assumption is not necessary.

- For technical reasons (in order to get the set B'), the Steinitz exchange lemma assumes that the sets A and B contain no repetitions.
  - Actually, it would be possible to state and prove a version of the Steinitz exchange lemma that allows repetitions.
  - However, this would be notationally messy.
- However, if we only care about the "$|A| \leq |B|$" part of the Steinitz exchange lemma (which is what we usually care about), then this assumption is not necessary.
- Indeed, suppose that V is a vector space over a field $\mathbb{F}$, and suppose that A is a linearly independent set of vectors in V and that B is a spanning set of V (with repetitions allowed).

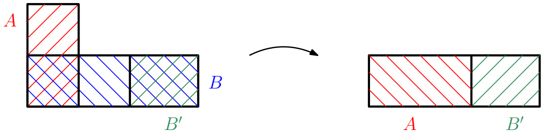
- For technical reasons (in order to get the set B'), the Steinitz exchange lemma assumes that the sets A and B contain no repetitions.
  - Actually, it would be possible to state and prove a version of the Steinitz exchange lemma that allows repetitions.
  - However, this would be notationally messy.
- However, if we only care about the "$|A| \leq |B|$" part of the Steinitz exchange lemma (which is what we usually care about), then this assumption is not necessary.
- Indeed, suppose that V is a vector space over a field $\mathbb{F}$, and suppose that A is a linearly independent set of vectors in V and that B is a spanning set of V (with repetitions allowed).
- Since A is linearly independent, it contains no repetitions; however, B may possibly contain repetitions.

- For technical reasons (in order to get the set B'), the Steinitz exchange lemma assumes that the sets A and B contain no repetitions.
  - Actually, it would be possible to state and prove a version of the Steinitz exchange lemma that allows repetitions.
  - However, this would be notationally messy.
- However, if we only care about the "$|A| \leq |B|$" part of the Steinitz exchange lemma (which is what we usually care about), then this assumption is not necessary.
- Indeed, suppose that V is a vector space over a field $\mathbb{F}$, and suppose that A is a linearly independent set of vectors in V and that B is a spanning set of V (with repetitions allowed).
- Since A is linearly independent, it contains no repetitions; however, B may possibly contain repetitions.
- But then we let $\widetilde{B}$ be the set obtained from B by eliminating repetitions.

- For technical reasons (in order to get the set B'), the Steinitz exchange lemma assumes that the sets A and B contain no repetitions.
  - Actually, it would be possible to state and prove a version of the Steinitz exchange lemma that allows repetitions.
  - However, this would be notationally messy.
- However, if we only care about the "$|A| \leq |B|$" part of the Steinitz exchange lemma (which is what we usually care about), then this assumption is not necessary.
- Indeed, suppose that V is a vector space over a field $\mathbb{F}$, and suppose that A is a linearly independent set of vectors in V and that B is a spanning set of V (with repetitions allowed).
- Since A is linearly independent, it contains no repetitions; however, B may possibly contain repetitions.
- But then we let $\widetilde{B}$ be the set obtained from B by eliminating repetitions.
- Then $\widetilde{B}$ is still a spanning set of V, and by the Steinitz exchange lemma, we get that $|A| \leq |\widetilde{B}| \leq |B|$.

The Steinitz exchange lemma

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and assume that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then $k \leq \ell$ (i.e. $|A| \leq |B|$). Moreover, there exists a set $B' \subseteq B \setminus A$ s.t. $|B'| = |B| - |A| = \ell - k$ and $A \cup B'$ is a spanning set of V.



- The most important corollary of the Steinitz exchange lemma is Theorem 3.2.16 (next slide).
- We first prove Theorem 3.2.16 (using the Steinitz exchange lemma), and then we prove the Steinitz exchange lemma.

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Proof (assuming the Steinitz exchange lemma).

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Proof (assuming the Steinitz exchange lemma). We apply the Steinitz exchange lemma twice.

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Proof (assuming the Steinitz exchange lemma). We apply the Steinitz exchange lemma twice.

Fix bases $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V.

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Proof (assuming the Steinitz exchange lemma). We apply the Steinitz exchange lemma twice.

Fix bases $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V.

Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is linearly independent and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a spanning set of V, the Steinitz exchange lemma guarantees that $m \leq n$.

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Proof (assuming the Steinitz exchange lemma). We apply the Steinitz exchange lemma twice.

Fix bases $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V.

Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is linearly independent and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a spanning set of V, the Steinitz exchange lemma guarantees that $m \leq n$.

On the other hand, since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set and $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a spanning set of V, the Steinitz exchange lemma guarantees that $n \leq m$.

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Proof (assuming the Steinitz exchange lemma). We apply the Steinitz exchange lemma twice.

Fix bases $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of V.

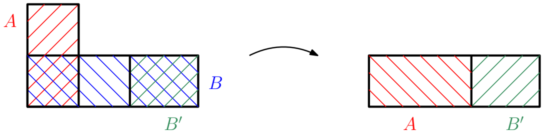
Since $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is linearly independent and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a spanning set of V, the Steinitz exchange lemma guarantees that $m \leq n$.

On the other hand, since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a linearly independent set and $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a spanning set of V, the Steinitz exchange lemma guarantees that $n \leq m$.

So, $m = n$. $\square$

The Steinitz exchange lemma

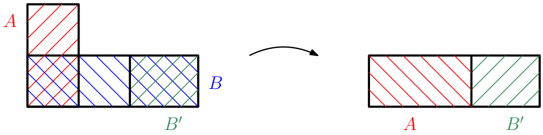
Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and assume that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then $k \leq \ell$ (i.e. $|A| \leq |B|$). Moreover, there exists a set $B' \subseteq B \setminus A$ s.t. $|B'| = |B| - |A| = \ell - k$ and $A \cup B'$ is a spanning set of V.



- Let's prove the Steinitz exchange lemma!

The Steinitz exchange lemma

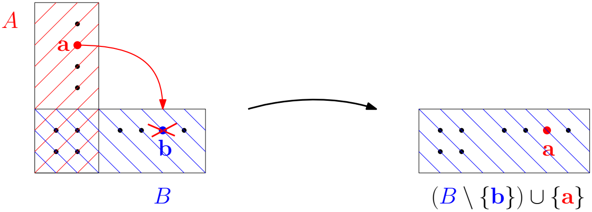
Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and assume that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then $k \leq \ell$ (i.e. $|A| \leq |B|$). Moreover, there exists a set $B' \subseteq B \setminus A$ s.t. $|B'| = |B| - |A| = \ell - k$ and $A \cup B'$ is a spanning set of V.



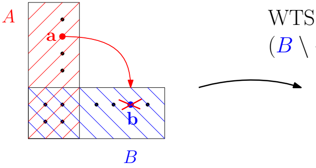
- Let's prove the Steinitz exchange lemma!
- The proof proceeds by induction using the following lemma (next slide).

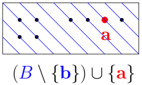
Lemma 3.2.15

Let V be a vector space over a field $\mathbb{F}$. Let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then for all $\mathbf{a} \in A \setminus B$, there exists some $\mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V.

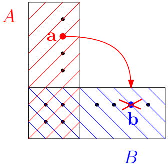


WTS $\forall \mathbf{a} \in A \setminus B \ \ \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V

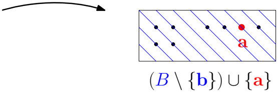




Proof.

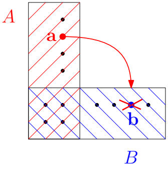


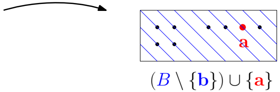
WTS $\forall \mathbf{a} \in A \setminus B \;\; \exists \mathbf{b} \in B \setminus A$ s.t.
$(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V



Proof. WMA $A \not\subseteq B$, for otherwise, the lemma is vacuously true.

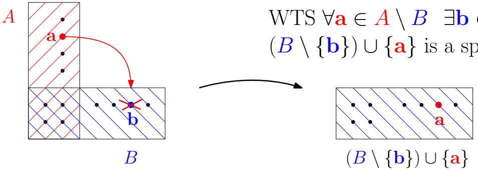
WTS $\forall \mathbf{a} \in A \setminus B \quad \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V





Proof. WMA $A \not\subseteq B$, for otherwise, the lemma is vacuously true. Fix any $\mathbf{a} \in A \setminus B$.

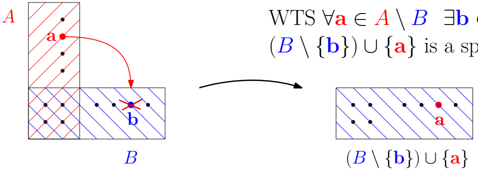
WTS $\forall \mathbf{a} \in A \setminus B \;\; \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V



Proof. WMA $A \not\subseteq B$, for otherwise, the lemma is vacuously true. Fix any $\mathbf{a} \in A \setminus B$. Then there exists an index $i \in \{1, \dots, k\}$ s.t. $\mathbf{a} = \mathbf{a}_i$. Since $\mathbf{a}_i \in V = \text{Span}(B)$, we know that there exist scalars $\alpha_1, \dots, \alpha_\ell \in \mathbb{F}$ s.t.

$$\mathbf{a}_i \;\; = \;\; \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell.$$

WTS $\forall \mathbf{a} \in A \setminus B \;\; \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V

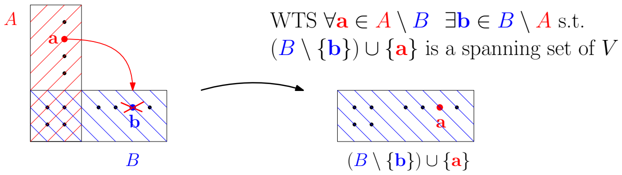


Proof. WMA $A \not\subseteq B$, for otherwise, the lemma is vacuously true. Fix any $\mathbf{a} \in A \setminus B$. Then there exists an index $i \in \{1, \dots, k\}$ s.t. $\mathbf{a} = \mathbf{a}_i$. Since $\mathbf{a}_i \in V = \text{Span}(B)$, we know that there exist scalars $\alpha_1, \dots, \alpha_\ell \in \mathbb{F}$ s.t.

$$\mathbf{a}_i \quad = \quad \alpha_1 \mathbf{b}_1 + \dots + \alpha_\ell \mathbf{b}_\ell.$$

> ***Claim.*** *There exists an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.*

Proof of the Claim.

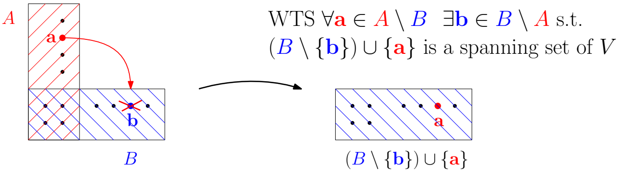


Proof. WMA $A \not\subseteq B$, for otherwise, the lemma is vacuously true. Fix any $\mathbf{a} \in A \setminus B$. Then there exists an index $i \in \{1, \dots, k\}$ s.t. $\mathbf{a} = \mathbf{a}_i$. Since $\mathbf{a}_i \in V = \text{Span}(B)$, we know that there exist scalars $\alpha_1, \dots, \alpha_\ell \in \mathbb{F}$ s.t.

$$\mathbf{a}_i \quad = \quad \alpha_1 \mathbf{b}_1 + \dots + \alpha_\ell \mathbf{b}_\ell.$$

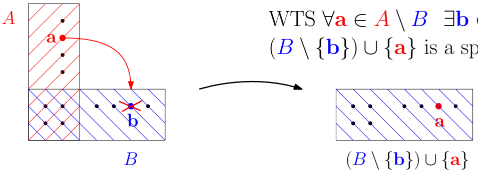
Claim. *There exists an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.*

Proof of the Claim. Suppose otherwise. Then for all $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$, we have that $\mathbf{b}_j \in B \cap A \subseteq A \setminus \{\mathbf{a}_i\}$. But now $\mathbf{a}_i$ is a linear combination of the other vectors in the linearly independent set A, contrary to Proposition 3.2.11(a). ♦



Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$.

Claim. *There exists an index $j \in \{1, \ldots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.*

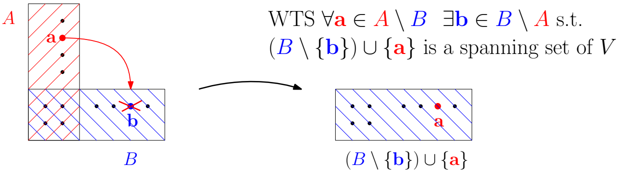


WTS $\forall \mathbf{a} \in A \setminus B \;\; \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V

Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$.

> ***Claim.*** *There exists an index $j \in \{1, \ldots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.*

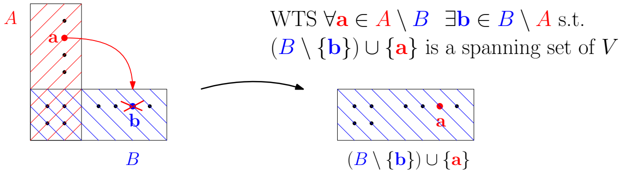
Using the Claim, we fix an index $j \in \{1, \ldots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.



Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$.

> ***Claim.*** *There exists an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.*

Using the Claim, we fix an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$. We will show that $(B \setminus \{\mathbf{b}_j\}) \cup \{\mathbf{a}_i\}$ is a spanning set of V (this will complete the proof of the lemma).



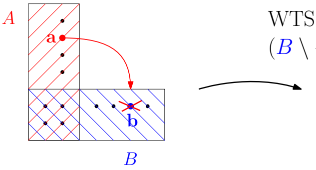
Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$.

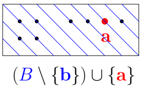
> ***Claim.*** *There exists an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.*

Using the Claim, we fix an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$. We will show that $(B \setminus \{\mathbf{b}_j\}) \cup \{\mathbf{a}_i\}$ is a spanning set of V (this will complete the proof of the lemma).

Since $\mathbf{b}_j \neq \mathbf{a}_i$, we see that $(B \setminus \{\mathbf{b}_j\}) \cup \{\mathbf{a}_i\} = (B \cup \{\mathbf{a}_i\}) \setminus \{\mathbf{b}_j\}$,

WTS $\forall \mathbf{a} \in A \setminus B \;\; \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V



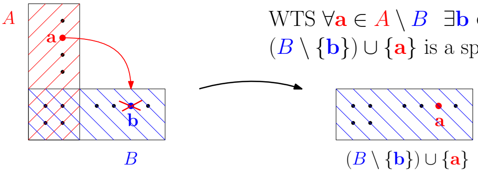


Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \dots + \alpha_\ell \mathbf{b}_\ell$.

> ***Claim.*** *There exists an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.*

Using the Claim, we fix an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$. We will show that $(B \setminus \{\mathbf{b}_j\}) \cup \{\mathbf{a}_i\}$ is a spanning set of V (this will complete the proof of the lemma).

Since $\mathbf{b}_j \neq \mathbf{a}_i$, we see that $(B \setminus \{\mathbf{b}_j\}) \cup \{\mathbf{a}_i\} = (B \cup \{\mathbf{a}_i\}) \setminus \{\mathbf{b}_j\}$, and we need to show that $(B \cup \{\mathbf{a}_i\}) \setminus \{\mathbf{b}_j\}$ is a spanning set of V.



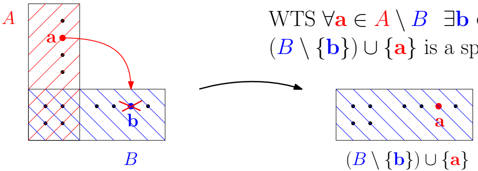
WTS $\forall \mathbf{a} \in A \setminus B \ \ \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V

Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \dots + \alpha_\ell \mathbf{b}_\ell$.

> ***Claim.*** *There exists an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.*

Using the Claim, we fix an index $j \in \{1, \dots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$. We will show that $(B \setminus \{\mathbf{b}_j\}) \cup \{\mathbf{a}_i\}$ is a spanning set of V (this will complete the proof of the lemma).

Since $\mathbf{b}_j \neq \mathbf{a}_i$, we see that $(B \setminus \{\mathbf{b}_j\}) \cup \{\mathbf{a}_i\} = (B \cup \{\mathbf{a}_i\}) \setminus \{\mathbf{b}_j\}$, and we need to show that $(B \cup \{\mathbf{a}_i\}) \setminus \{\mathbf{b}_j\}$ is a spanning set of V. Since B is a spanning set of V, so is $B \cup \{\mathbf{a}_i\}$.



WTS $\forall \mathbf{a} \in A \setminus B \ \ \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V

Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$.

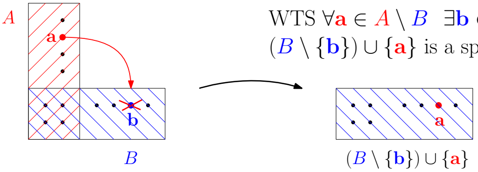
> ***Claim.*** *There exists an index $j \in \{1, \ldots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$.*

Using the Claim, we fix an index $j \in \{1, \ldots, \ell\}$ s.t. $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$. We will show that $(B \setminus \{\mathbf{b}_j\}) \cup \{\mathbf{a}_i\}$ is a spanning set of V (this will complete the proof of the lemma).

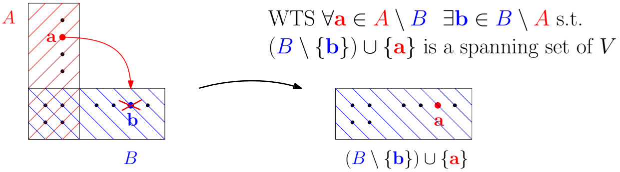
Since $\mathbf{b}_j \neq \mathbf{a}_i$, we see that $(B \setminus \{\mathbf{b}_j\}) \cup \{\mathbf{a}_i\} = (B \cup \{\mathbf{a}_i\}) \setminus \{\mathbf{b}_j\}$, and we need to show that $(B \cup \{\mathbf{a}_i\}) \setminus \{\mathbf{b}_j\}$ is a spanning set of V. Since B is a spanning set of V, so is $B \cup \{\mathbf{a}_i\}$.

In view of Proposition 3.2.11(b), it now suffices to show that $\mathbf{b}_j$ is a linear combination of the other vectors in $B \cup \{\mathbf{a}_i\}$.

WTS $\forall \mathbf{a} \in A \setminus B \ \ \exists \mathbf{b} \in B \setminus A$ s.t.
$(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V



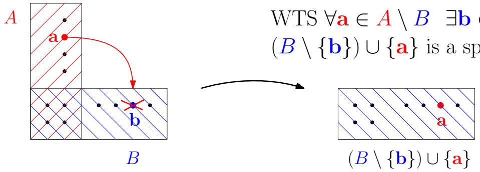
Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$; $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$. WTS $\mathbf{b}_j$ is a linear combination of the other vectors in $B \cup \{\mathbf{a}_i\}$.



Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$; $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$. WTS $\mathbf{b}_j$ is a linear combination of the other vectors in $B \cup \{\mathbf{a}_i\}$.

Since $\mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$, we see that

$$\alpha_j \mathbf{b}_j = \mathbf{a}_i - \alpha_1 \mathbf{b}_1 - \cdots - \alpha_{j-1} \mathbf{b}_{j-1} - \alpha_{j+1} \mathbf{b}_{j+1} - \cdots - \alpha_\ell \mathbf{b}_\ell.$$



WTS $\forall \mathbf{a} \in A \setminus B \ \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V

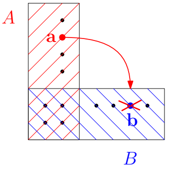
Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$; $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$. WTS $\mathbf{b}_j$ is a linear combination of the other vectors in $B \cup \{\mathbf{a}_i\}$.

Since $\mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$, we see that

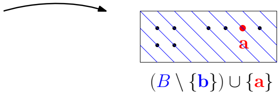
$$\alpha_j \mathbf{b}_j = \mathbf{a}_i - \alpha_1 \mathbf{b}_1 - \cdots - \alpha_{j-1} \mathbf{b}_{j-1} - \alpha_{j+1} \mathbf{b}_{j+1} - \cdots - \alpha_\ell \mathbf{b}_\ell.$$

Since $\alpha_j \neq 0$, we know that α_j has a multiplicative inverse α_j^{-1}, and we deduce that

$$\begin{aligned} \mathbf{b}_j = \; & \alpha_j^{-1} \mathbf{a}_i - \alpha_j^{-1} \alpha_1 \mathbf{b}_1 - \cdots - \alpha_j^{-1} \alpha_{j-1} \mathbf{b}_{j-1} - \\ & - \alpha_j^{-1} \alpha_{j+1} \mathbf{b}_{j+1} - \cdots - \alpha_j^{-1} \alpha_\ell \mathbf{b}_\ell. \end{aligned}$$



WTS $\forall \mathbf{a} \in A \setminus B \;\; \exists \mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V



Proof (continued). Reminder: $\mathbf{a} = \mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$; $\alpha_j \neq 0$ and $\mathbf{b}_j \in B \setminus A$. WTS $\mathbf{b}_j$ is a linear combination of the other vectors in $B \cup \{\mathbf{a}_i\}$.

Since $\mathbf{a}_i = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_\ell \mathbf{b}_\ell$, we see that

$$\alpha_j \mathbf{b}_j = \mathbf{a}_i - \alpha_1 \mathbf{b}_1 - \cdots - \alpha_{j-1} \mathbf{b}_{j-1} - \alpha_{j+1} \mathbf{b}_{j+1} - \cdots - \alpha_\ell \mathbf{b}_\ell.$$

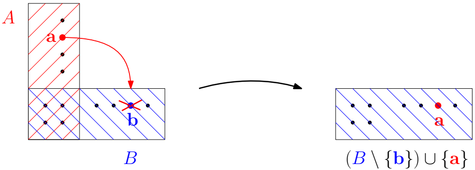
Since $\alpha_j \neq 0$, we know that α_j has a multiplicative inverse α_j^{-1}, and we deduce that

$$\begin{aligned} \mathbf{b}_j &= \alpha_j^{-1} \mathbf{a}_i - \alpha_j^{-1} \alpha_1 \mathbf{b}_1 - \cdots - \alpha_j^{-1} \alpha_{j-1} \mathbf{b}_{j-1} - \\ &\quad - \alpha_j^{-1} \alpha_{j+1} \mathbf{b}_{j+1} - \cdots - \alpha_j^{-1} \alpha_\ell \mathbf{b}_\ell. \end{aligned}$$

So, $\mathbf{b}_j$ is indeed a linear combination of the other vectors in $B \cup \{\mathbf{a}_i\}$, and we are done. $\square$

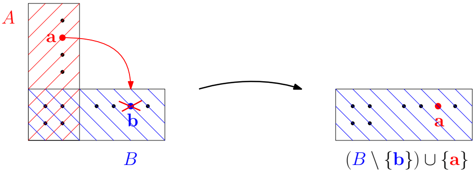
Lemma 3.2.15

Let V be a vector space over a field $\mathbb{F}$. Let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then for all $\mathbf{a} \in A \setminus B$, there exists some $\mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V.



Lemma 3.2.15

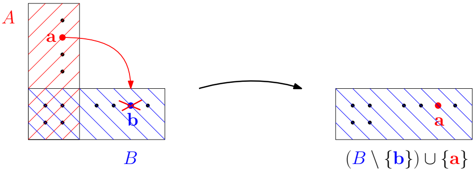
Let V be a vector space over a field $\mathbb{F}$. Let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then for all $\mathbf{a} \in A \setminus B$, there exists some $\mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V.



- The proof of the Steinitz exchange lemma consists of repeated applications of Lemma 3.2.15 (technically, an induction).

Lemma 3.2.15

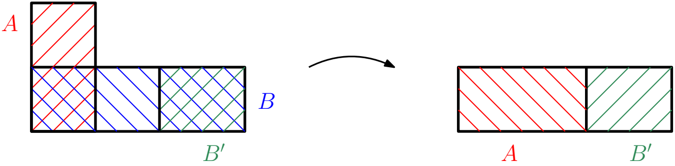
Let V be a vector space over a field $\mathbb{F}$. Let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then for all $\mathbf{a} \in A \setminus B$, there exists some $\mathbf{b} \in B \setminus A$ s.t. $(B \setminus \{\mathbf{b}\}) \cup \{\mathbf{a}\}$ is a spanning set of V.

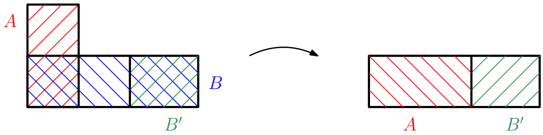


- The proof of the Steinitz exchange lemma consists of repeated applications of Lemma 3.2.15 (technically, an induction).
  - The formal proof is in the Lecture Notes.
  - Here, we give an informal outline.

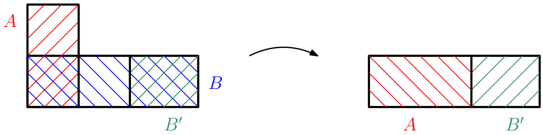
The Steinitz exchange lemma

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell \in V$, and assume that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are pairwise distinct and that $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ are pairwise distinct. Assume furthermore that $A := \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a linearly independent set in V, and assume that $B := \{\mathbf{b}_1, \dots, \mathbf{b}_\ell\}$ is a spanning set of V. Then $k \leq \ell$ (i.e. $|A| \leq |B|$). Moreover, there exists a set $B' \subseteq B \setminus A$ s.t. $|B'| = |B| - |A| = \ell - k$ and $A \cup B'$ is a spanning set of V.

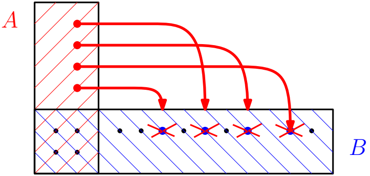




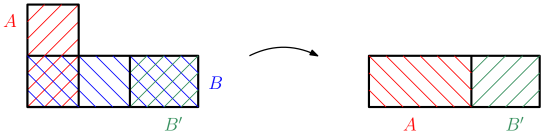
Proof (outline).



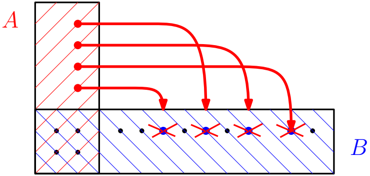
Proof (outline). Using Lemma 3.2.15, we "throw in" vertices of $A \setminus B$ into B one by one, and at each step, we remove one vertex of $B \setminus A$ from B.



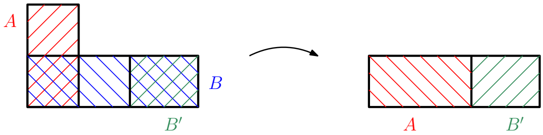
By Lemma 3.2.15, at each step, the set that we create remains a spanning set of V.



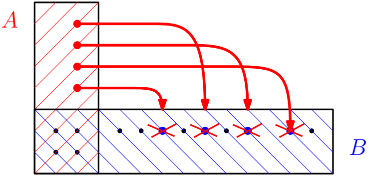
Proof (outline). Using Lemma 3.2.15, we “throw in” vertices of $A \setminus B$ into B one by one, and at each step, we remove one vertex of $B \setminus A$ from B.



By Lemma 3.2.15, at each step, the set that we create remains a spanning set of V. In the end, we obtain a spanning set of V that includes A (as a subset) and is of the same set as B.



Proof (outline). Using Lemma 3.2.15, we "throw in" vertices of $A \setminus B$ into B one by one, and at each step, we remove one vertex of $B \setminus A$ from B.



By Lemma 3.2.15, at each step, the set that we create remains a spanning set of V. In the end, we obtain a spanning set of V that includes A (as a subset) and is of the same set as B.

B' is the set of all vertices of $B \setminus A$ that we did not "throw out" in the process. □

- Reminder: Using the Steinitz exchange lemma, we proved Theorem 3.2.16 (below).

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

- Reminder: Using the Steinitz exchange lemma, we proved Theorem 3.2.16 (below).

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Definition

The *dimension* of a finite-dimensional vector space V over a field $\mathbb{F}$, denoted by $\dim(V)$, is the number of elements in any basis of V (by Theorem 3.2.16, this is well-defined).

- Reminder: Using the Steinitz exchange lemma, we proved Theorem 3.2.16 (below).

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Definition

The *dimension* of a finite-dimensional vector space V over a field $\mathbb{F}$, denoted by $\dim(V)$, is the number of elements in any basis of V (by Theorem 3.2.16, this is well-defined).

- **Remarks:**

- Reminder: Using the Steinitz exchange lemma, we proved Theorem 3.2.16 (below).

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Definition

The *dimension* of a finite-dimensional vector space V over a field $\mathbb{F}$, denoted by $\dim(V)$, is the number of elements in any basis of V (by Theorem 3.2.16, this is well-defined).

- **Remarks:**
  - Note that $\dim(\{\mathbf{0}\}) = 0$ (where $\{\mathbf{0}\}$ is understood to be a vector space over an arbitrary field $\mathbb{F}$), because $\emptyset$ is a basis of $\{\mathbf{0}\}$.

- Reminder: Using the Steinitz exchange lemma, we proved Theorem 3.2.16 (below).

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Definition

The *dimension* of a finite-dimensional vector space V over a field $\mathbb{F}$, denoted by $\dim(V)$, is the number of elements in any basis of V (by Theorem 3.2.16, this is well-defined).

- **Remarks:**
  - Note that $\dim(\{\mathbf{0}\}) = 0$ (where $\{\mathbf{0}\}$ is understood to be a vector space over an arbitrary field $\mathbb{F}$), because $\emptyset$ is a basis of $\{\mathbf{0}\}$.
  - For any field $\mathbb{F}$, we have that $\dim(\mathbb{F}^n) = n$, because the standard basis of $\mathbb{F}^n$ has n elements.
    - However, the standard basis is not the only basis of $\mathbb{F}^n$ (except in some very special cases).

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

Proof.

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

Proof. Fix a basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of V.

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

Proof. Fix a basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of V. Then B is both a linearly independent set and a spanning set of V.

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

Proof. Fix a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of V. Then B is both a linearly independent set and a spanning set of V.

Now, by the Steinitz exchange lemma, the number of vectors in any linearly independent set of V is at most the number of vectors in the spanning set B of V, which is n; so, (a) holds.

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

Proof. Fix a basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of V. Then B is both a linearly independent set and a spanning set of V.

Now, by the Steinitz exchange lemma, the number of vectors in any linearly independent set of V is at most the number of vectors in the spanning set B of V, which is n; so, (a) holds.

On the other hand, by the Steinitz exchange lemma, any spanning set of V has at least as many vectors as the linearly independent set B; so, (b) holds. $\square$

Theorem 3.2.16

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Then all bases of V are of the same size.

Definition

The *dimension* of a finite-dimensional vector space V over a field $\mathbb{F}$, denoted by $\dim(V)$, is the number of elements in any basis of V (by Theorem 3.2.16, this is well-defined).

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

- Informally, Theorem 3.2.17 says:

$$|\text{linearly independent set of } V| \quad \leq \quad \dim(V) \quad \leq \quad |\text{spanning set of } V|.$$

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

- By Theorem 3.2.17(a), linearly independent sets in any finite-dimensional vector space have bounded size (bounded above by the dimension of the vector space in question).

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

- By Theorem 3.2.17(a), linearly independent sets in any finite-dimensional vector space have bounded size (bounded above by the dimension of the vector space in question).
- On the other hand, by Proposition 3.2.18 (next slide), infinite-dimensional vector spaces have linearly independent sets of arbitrarily large (finite) size.

Theorem 3.2.17

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) every linearly independent set of vectors in V has at most n vectors;

(b) every spanning set of V has at least n vectors.

- By Theorem 3.2.17(a), linearly independent sets in any finite-dimensional vector space have bounded size (bounded above by the dimension of the vector space in question).
- On the other hand, by Proposition 3.2.18 (next slide), infinite-dimensional vector spaces have linearly independent sets of arbitrarily large (finite) size.
  - For instance, if $\mathbb{F}$ is a field, then for any positive integer n, $\{1, x, x^2, \dots, x^n\}$ is a linearly independent set in $\mathbb{P}_{\mathbb{F}}$ (the vector space of all polynomials with coefficients in $\mathbb{F}$).

Proposition 3.2.18

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Then for every non-negative integer n, V has a linearly independent set of size n.

Proof (outline).

Proposition 3.2.18

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Then for every non-negative integer n, V has a linearly independent set of size n.

Proof (outline). We proceed by induction on n.

Proposition 3.2.18

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Then for every non-negative integer n, V has a linearly independent set of size n.

Proof (outline). We proceed by induction on n.

For $n = 0$, we observe that $\emptyset$ is a linearly independent set of size 0 in V.

Proposition 3.2.18

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Then for every non-negative integer n, V has a linearly independent set of size n.

Proof (outline). We proceed by induction on n.

For $n = 0$, we observe that $\emptyset$ is a linearly independent set of size 0 in V.

Next, fix a non-negative integer n, and assume that V has a linearly independent set of size n, say $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$.

Proposition 3.2.18

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Then for every non-negative integer n, V has a linearly independent set of size n.

Proof (outline). We proceed by induction on n.

For $n = 0$, we observe that $\emptyset$ is a linearly independent set of size 0 in V.

Next, fix a non-negative integer n, and assume that V has a linearly independent set of size n, say $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Then $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is not a spanning set of V, for otherwise, it would be a basis of V, contrary to the fact that V is infinite-dimensional.

Proposition 3.2.18

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Then for every non-negative integer n, V has a linearly independent set of size n.

Proof (outline). We proceed by induction on n.

For $n = 0$, we observe that $\emptyset$ is a linearly independent set of size 0 in V.

Next, fix a non-negative integer n, and assume that V has a linearly independent set of size n, say $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Then $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is not a spanning set of V, for otherwise, it would be a basis of V, contrary to the fact that V is infinite-dimensional. Thus, $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n) \subsetneqq V$;

Proposition 3.2.18

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Then for every non-negative integer n, V has a linearly independent set of size n.

Proof (outline). We proceed by induction on n.

For $n = 0$, we observe that $\emptyset$ is a linearly independent set of size 0 in V.

Next, fix a non-negative integer n, and assume that V has a linearly independent set of size n, say $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Then $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is not a spanning set of V, for otherwise, it would be a basis of V, contrary to the fact that V is infinite-dimensional. Thus, $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n) \subsetneqq V$; fix some $\mathbf{a}_{n+1} \in V \setminus \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$.

Proposition 3.2.18

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Then for every non-negative integer n, V has a linearly independent set of size n.

Proof (outline). We proceed by induction on n.

For $n = 0$, we observe that $\emptyset$ is a linearly independent set of size 0 in V.

Next, fix a non-negative integer n, and assume that V has a linearly independent set of size n, say $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Then $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is not a spanning set of V, for otherwise, it would be a basis of V, contrary to the fact that V is infinite-dimensional. Thus, $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n) \subsetneqq V$; fix some $\mathbf{a}_{n+1} \in V \setminus \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$. Then $\{\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{a}_{n+1}\}$ is a linearly independent set in V (details: Lecture Notes).

Proposition 3.2.18

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Then for every non-negative integer n, V has a linearly independent set of size n.

Proof (outline). We proceed by induction on n.

For $n = 0$, we observe that $\emptyset$ is a linearly independent set of size 0 in V.

Next, fix a non-negative integer n, and assume that V has a linearly independent set of size n, say $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Then $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is not a spanning set of V, for otherwise, it would be a basis of V, contrary to the fact that V is infinite-dimensional. Thus, $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n) \subsetneqq V$; fix some $\mathbf{a}_{n+1} \in V \setminus \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$. Then $\{\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{a}_{n+1}\}$ is a linearly independent set in V (details: Lecture Notes). This completes the induction. $\square$

- Reminder:

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

- Reminder:

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

- For linearly independent sets, we have the following analog of Theorem 3.2.14:

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

- We first make some remarks and then give a proof.

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

- Suppose that V is a vector space over a field $\mathbb{F}$.

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

- Suppose that V is a vector space over a field $\mathbb{F}$.
  - By Theorem 3.2.14, any (finite) spanning set of V contains a subset that is a basis of V; in particular, if a vector space has a (finite) spanning set, then it is finite-dimensional.

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

- Suppose that V is a vector space over a field $\mathbb{F}$.
  - By Theorem 3.2.14, any (finite) spanning set of V contains a subset that is a basis of V; in particular, if a vector space has a (finite) spanning set, then it is finite-dimensional.
  - On the other hand, by Theorem 3.2.19, if V is **finite-dimensional**, then any linearly independent set in V can be extended to a basis of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \ldots, \mathbf{a}_k$.

Proof. Set $n := \dim(V)$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof. Set $n := \dim(V)$. By Theorem 3.2.17, any linearly independent set of vectors in V has at most n vectors; in particular, $k \leq n$ (because $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is linearly independent).

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \ldots, \mathbf{a}_k$.

Proof. Set $n := \dim(V)$. By Theorem 3.2.17, any linearly independent set of vectors in V has at most n vectors; in particular, $k \leq n$ (because $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ is linearly independent).

Now, let A be a linearly independent set that contains vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

- Let us explain why A exists.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof. Set $n := \dim(V)$. By Theorem 3.2.17, any linearly independent set of vectors in V has at most n vectors; in particular, $k \leq n$ (because $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is linearly independent).

Now, let A be a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

- Let us explain why A exists.
- There exists at least one linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, namely, the set $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof. Set $n := \dim(V)$. By Theorem 3.2.17, any linearly independent set of vectors in V has at most n vectors; in particular, $k \leq n$ (because $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is linearly independent).

Now, let A be a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

- Let us explain why A exists.
- There exists at least one linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, namely, the set $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$.
- On the other hand, all linearly independent sets are of size at most n, and in particular, there is an upper bound on the size of linearly independent sets containing $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof. Set $n := \dim(V)$. By Theorem 3.2.17, any linearly independent set of vectors in V has at most n vectors; in particular, $k \leq n$ (because $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is linearly independent).

Now, let A be a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

- Let us explain why A exists.
- There exists at least one linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, namely, the set $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$.
- On the other hand, all linearly independent sets are of size at most n, and in particular, there is an upper bound on the size of linearly independent sets containing $\mathbf{a}_1, \dots, \mathbf{a}_k$.
- So, A exists.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: A is a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: A is a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

Set $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: A is a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

Set $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$. WTS is a basis of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: A is a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

Set $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$. WTS is a basis of V.

Since A is linearly independent, it suffices to show that A is a spanning set of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: A is a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

Set $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$. WTS is a basis of V.

Since A is linearly independent, it suffices to show that A is a spanning set of V. Fix $\mathbf{v} \in V$; WTS $\mathbf{v}$ is a linear combination of vectors in A.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: A is a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

Set $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$. WTS is a basis of V.

Since A is linearly independent, it suffices to show that A is a spanning set of V. Fix $\mathbf{v} \in V$; WTS $\mathbf{v}$ is a linear combination of vectors in A. If $\mathbf{v} \in A$, then this is immediate. So, assume that $\mathbf{v} \notin A$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: A is a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

Set $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$. WTS is a basis of V.

Since A is linearly independent, it suffices to show that A is a spanning set of V. Fix $\mathbf{v} \in V$; WTS $\mathbf{v}$ is a linear combination of vectors in A. If $\mathbf{v} \in A$, then this is immediate. So, assume that $\mathbf{v} \notin A$. Then by the maximality of A, the set $\{\mathbf{v}\} \cup A$ is not linearly independent.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: A is a linearly independent set that contains vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$, and subject to that, is of maximum possible size.

Set $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$. WTS is a basis of V.

Since A is linearly independent, it suffices to show that A is a spanning set of V. Fix $\mathbf{v} \in V$; WTS $\mathbf{v}$ is a linear combination of vectors in A. If $\mathbf{v} \in A$, then this is immediate. So, assume that $\mathbf{v} \notin A$. Then by the maximality of A, the set $\{\mathbf{v}\} \cup A$ is not linearly independent. So, there exist scalars $\alpha_0, \alpha_1, \dots, \alpha_{k+\ell} \in \mathbb{F}$, not all zero, s.t.

$$\alpha_0 \mathbf{v} + \alpha_1 \mathbf{a}_1 + \dots + \alpha_{k+\ell} \mathbf{a}_{k+\ell} \quad = \quad \mathbf{0}.$$

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: $\alpha_0 \mathbf{v} + \alpha_1 \mathbf{a}_1 + \dots + \alpha_{k+\ell} \mathbf{a}_{k+\ell} = \mathbf{0}$. WTS $\mathbf{v}$ is a linear combination of the vectors in $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: $\alpha_0\mathbf{v} + \alpha_1\mathbf{a}_1 + \dots + \alpha_{k+\ell}\mathbf{a}_{k+\ell} = \mathbf{0}$. WTS $\mathbf{v}$ is a linear combination of the vectors in $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$.

If $\alpha_0 = 0$, then at least one of $\alpha_1, \dots, \alpha_{k+\ell}$ is non-zero and $\alpha_1\mathbf{a}_1 + \dots + \alpha_{k+\ell}\mathbf{a}_{k+\ell} = \mathbf{0}$, contrary to the fact that A is linearly independent.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Proof (continued). Reminder: $\alpha_0 \mathbf{v} + \alpha_1 \mathbf{a}_1 + \dots + \alpha_{k+\ell} \mathbf{a}_{k+\ell} = \mathbf{0}$. WTS $\mathbf{v}$ is a linear combination of the vectors in $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_{k+\ell}\}$.

If $\alpha_0 = 0$, then at least one of $\alpha_1, \dots, \alpha_{k+\ell}$ is non-zero and $\alpha_1 \mathbf{a}_1 + \dots + \alpha_{k+\ell} \mathbf{a}_{k+\ell} = \mathbf{0}$, contrary to the fact that A is linearly independent.

So, $\alpha_0 \neq 0$, it follows that

$$\mathbf{v} = (-\alpha_0^{-1} \alpha_1) \mathbf{a}_1 + \dots + (-\alpha_0^{-1} \alpha_{k+\ell}) \mathbf{a}_{k+\ell},$$

and we see that $\mathbf{v}$ is a linear combination of vectors in A.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \ldots, \mathbf{a}_k$.

Proof (continued). Reminder: $\alpha_0\mathbf{v} + \alpha_1\mathbf{a}_1 + \cdots + \alpha_{k+\ell}\mathbf{a}_{k+\ell} = \mathbf{0}$.
WTS $\mathbf{v}$ is a linear combination of the vectors in
$A = \{\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_{k+\ell}\}$.

If $\alpha_0 = 0$, then at least one of $\alpha_1, \ldots, \alpha_{k+\ell}$ is non-zero and $\alpha_1\mathbf{a}_1 + \cdots + \alpha_{k+\ell}\mathbf{a}_{k+\ell} = \mathbf{0}$, contrary to the fact that A is linearly independent.

So, $\alpha_0 \neq 0$, it follows that

$$\mathbf{v} \;=\; (-\alpha_0^{-1}\alpha_1)\mathbf{a}_1 + \cdots + (-\alpha_0^{-1}\alpha_{k+\ell})\mathbf{a}_{k+\ell},$$

and we see that $\mathbf{v}$ is a linear combination of vectors in A.

This proves that A is a basis of V, and we are done. $\square$

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \ldots, \mathbf{a}_k$.

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

- Theorems 3.2.14 and 3.2.19 together yield the following corollary:

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) any linearly independent set of n vectors of V is a basis of V;

(b) any set of n vectors of V that spans V is a basis of V.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \ldots, \mathbf{a}_k$.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

- (a) any linearly independent set of n vectors of V is a basis of V;
- (b) any set of n vectors of V that spans V is a basis of V.

Proof of (a).

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) any linearly independent set of n vectors of V is a basis of V;

(b) any set of n vectors of V that spans V is a basis of V.

Proof of (a). Let A be any linearly independent set of vectors in V s.t. $|A| = n$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) any linearly independent set of n vectors of V is a basis of V;

(b) any set of n vectors of V that spans V is a basis of V.

Proof of (a). Let A be any linearly independent set of vectors in V s.t. $|A| = n$. By Theorem 3.2.19, V has a basis A' s.t. $A \subseteq A'$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) any linearly independent set of n vectors of V is a basis of V;

(b) any set of n vectors of V that spans V is a basis of V.

Proof of (a). Let A be any linearly independent set of vectors in V s.t. $|A| = n$. By Theorem 3.2.19, V has a basis A' s.t. $A \subseteq A'$. Since $\dim(V) = n$, we see that $|A'| = n$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

- (a) any linearly independent set of n vectors of V is a basis of V;
- (b) any set of n vectors of V that spans V is a basis of V.

Proof of (a). Let A be any linearly independent set of vectors in V s.t. $|A| = n$. By Theorem 3.2.19, V has a basis A' s.t. $A \subseteq A'$. Since $\dim(V) = n$, we see that $|A'| = n$. Since $|A| = n$ and $A \subseteq A'$, it follows that $A = A'$.

Theorem 3.2.19

Let V be a **finite-dimensional** vector space over a field $\mathbb{F}$, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly independent set of vectors in V. Then there exists some basis of V that contains all of $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) any linearly independent set of n vectors of V is a basis of V;

(b) any set of n vectors of V that spans V is a basis of V.

Proof of (a). Let A be any linearly independent set of vectors in V s.t. $|A| = n$. By Theorem 3.2.19, V has a basis A' s.t. $A \subseteq A'$. Since $\dim(V) = n$, we see that $|A'| = n$. Since $|A| = n$ and $A \subseteq A'$, it follows that $A = A'$. So, A is a basis of V (because A' is). This proves (a). $\square$

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

- (a) any linearly independent set of n vectors of V is a basis of V;
- (b) any set of n vectors of V that spans V is a basis of V.

Proof of (b).

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) any linearly independent set of n vectors of V is a basis of V;

(b) any set of n vectors of V that spans V is a basis of V.

Proof of (b). Let B be any set of n vectors of V s.t. $V = \text{Span}(B)$.

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) any linearly independent set of n vectors of V is a basis of V;

(b) any set of n vectors of V that spans V is a basis of V.

Proof of (b). Let B be any set of n vectors of V s.t. $V = \text{Span}(B)$. Then by Theorem 3.2.14, V has a basis B' s.t. $B' \subseteq B$.

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) any linearly independent set of n vectors of V is a basis of V;

(b) any set of n vectors of V that spans V is a basis of V.

Proof of (b). Let B be any set of n vectors of V s.t. $V = \text{Span}(B)$. Then by Theorem 3.2.14, V has a basis B' s.t. $B' \subseteq B$. Since $\dim(V) = n$, we see that $|B'| = n$.

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

(a) any linearly independent set of n vectors of V is a basis of V;

(b) any set of n vectors of V that spans V is a basis of V.

Proof of (b). Let B be any set of n vectors of V s.t. $V = \text{Span}(B)$. Then by Theorem 3.2.14, V has a basis B' s.t. $B' \subseteq B$. Since $\dim(V) = n$, we see that $|B'| = n$. Since $|B| = n$ and $B' \subseteq B$, it follows that $B' = B$.

Theorem 3.2.14

Let V be a vector space over a field $\mathbb{F}$, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a spanning set of V. Then some subset of B is a basis of V.

Corollary 3.2.20

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Then both the following hold:

- (a) any linearly independent set of n vectors of V is a basis of V;
- (b) any set of n vectors of V that spans V is a basis of V.

Proof of (b). Let B be any set of n vectors of V s.t. $V = \text{Span}(B)$. Then by Theorem 3.2.14, V has a basis B' s.t. $B' \subseteq B$. Since $\dim(V) = n$, we see that $|B'| = n$. Since $|B| = n$ and $B' \subseteq B$, it follows that $B' = B$. So, B is a basis of V (because B' is). This proves (b). $\square$

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

a. U is finite-dimensional;
b. $\dim(U) \leq \dim(V)$;
c. if $\dim(U) = \dim(V)$, then $U = V$.

Proof.

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof. Set $n := \dim(V)$.

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof. Set $n := \dim(V)$. Since U is a subspace in V, any linearly independent set of vectors in U is also linearly independent in V, and by Theorem 3.2.17(a), any such set contains at most n vectors.

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof. Set $n := \dim(V)$. Since U is a subspace in V, any linearly independent set of vectors in U is also linearly independent in V, and by Theorem 3.2.17(a), any such set contains at most n vectors. Now, let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set of vectors in U of maximum possible size. (Then $k \leq n$.)

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof. Set $n := \dim(V)$. Since U is a subspace in V, any linearly independent set of vectors in U is also linearly independent in V, and by Theorem 3.2.17(a), any such set contains at most n vectors. Now, let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set of vectors in U of maximum possible size. (Then $k \leq n$.) WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ spans U.

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof. Set $n := \dim(V)$. Since U is a subspace in V, any linearly independent set of vectors in U is also linearly independent in V, and by Theorem 3.2.17(a), any such set contains at most n vectors. Now, let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set of vectors in U of maximum possible size. (Then $k \leq n$.) WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ spans U.

- This will imply that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U, and consequently, that $\dim(U) = k \leq n$, which is enough to prove (a) and (b).

Proof (continued). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a linearly independent set in U of maximum possible size. WTS it spans U.

Proof (continued). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a linearly independent set in U of maximum possible size. WTS it spans U.

Fix $\mathbf{u} \in U$; WTS $\mathbf{u}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$.

Proof (continued). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a linearly independent set in U of maximum possible size. WTS it spans U.

Fix $\mathbf{u} \in U$; WTS $\mathbf{u}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$. If $\mathbf{u} \in \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, then this is immediate.

Proof (continued). Reminder: $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set in U of maximum possible size. WTS it spans U.

Fix $\mathbf{u} \in U$; WTS $\mathbf{u}$ is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_k$. If $\mathbf{u} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, then this is immediate. So, assume that $\mathbf{u} \notin \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$.

By the maximality of $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, we see that $\{\mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly dependent. So, there exist scalars $\alpha_0, \alpha_1, \dots, \alpha_k$, not all zero, s.t.

$$\alpha_0 \mathbf{u} + \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0}.$$

Proof (continued). Reminder: $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set in U of maximum possible size. WTS it spans U.

Fix $\mathbf{u} \in U$; WTS $\mathbf{u}$ is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_k$. If $\mathbf{u} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, then this is immediate. So, assume that $\mathbf{u} \notin \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$.

By the maximality of $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, we see that $\{\mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly dependent. So, there exist scalars $\alpha_0, \alpha_1, \dots, \alpha_k$, not all zero, s.t.

$$\alpha_0 \mathbf{u} + \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0}.$$

If $\alpha_0 = 0$, then $\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0}$ and at least one of the scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ is non-zero, contrary to the fact that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly independent.

Proof (continued). Reminder: $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set in U of maximum possible size. WTS it spans U.

Fix $\mathbf{u} \in U$; WTS $\mathbf{u}$ is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_k$. If $\mathbf{u} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, then this is immediate. So, assume that $\mathbf{u} \notin \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$.

By the maximality of $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, we see that $\{\mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly dependent. So, there exist scalars $\alpha_0, \alpha_1, \dots, \alpha_k$, not all zero, s.t.

$$\alpha_0 \mathbf{u} + \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0}.$$

If $\alpha_0 = 0$, then $\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0}$ and at least one of the scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ is non-zero, contrary to the fact that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly independent.

So, $\alpha_0 \neq 0$, and we deduce that

$$\mathbf{u} = (-\alpha_0^{-1}\alpha_1)\mathbf{u}_1 + \dots + (-\alpha_0^{-1}\alpha_k)\mathbf{u}_k.$$

Proof (continued). Reminder: $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set in U of maximum possible size. WTS it spans U.

Fix $\mathbf{u} \in U$; WTS $\mathbf{u}$ is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_k$. If $\mathbf{u} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, then this is immediate. So, assume that $\mathbf{u} \notin \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$.

By the maximality of $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, we see that $\{\mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly dependent. So, there exist scalars $\alpha_0, \alpha_1, \dots, \alpha_k$, not all zero, s.t.

$$\alpha_0\mathbf{u} + \alpha_1\mathbf{u}_1 + \dots + \alpha_k\mathbf{u}_k \quad = \quad \mathbf{0}.$$

If $\alpha_0 = 0$, then $\alpha_1\mathbf{u}_1 + \dots + \alpha_k\mathbf{u}_k = \mathbf{0}$ and at least one of the scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ is non-zero, contrary to the fact that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly independent.

So, $\alpha_0 \neq 0$, and we deduce that

$$\mathbf{u} = (-\alpha_0^{-1}\alpha_1)\mathbf{u}_1 + \dots + (-\alpha_0^{-1}\alpha_k)\mathbf{u}_k.$$

So, $\mathbf{u} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$, and we deduce that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a spanning set of U.

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof (continued). We have now shown that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U, and consequently, (a) and (b) hold.

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof (continued). We have now shown that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U, and consequently, (a) and (b) hold.

It remains to prove (c).

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof (continued). We have now shown that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U, and consequently, (a) and (b) hold.

It remains to prove (c). Suppose that $\dim(U) = \dim(V)$, i.e. $k = n$.

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof (continued). We have now shown that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U, and consequently, (a) and (b) hold.

It remains to prove (c). Suppose that $\dim(U) = \dim(V)$, i.e. $k = n$.

But now $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set of n vectors in V, and so Corollary 3.2.20 guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of V.

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

Proof (continued). We have now shown that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U, and consequently, (a) and (b) hold.

It remains to prove (c). Suppose that $\dim(U) = \dim(V)$, i.e. $k = n$.

But now $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set of n vectors in V, and so Corollary 3.2.20 guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of V. So, $U = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k) = V$, and we are done. $\square$

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

- **Warning:** Theorem 3.2.21(c) fails if V is infinite-dimensional!

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

(c) if $\dim(U) = \dim(V)$, then $U = V$.

- **Warning:** Theorem 3.2.21(c) fails if V is infinite-dimensional!
  - Infinite-dimensional vector spaces can have proper subspaces that are infinite-dimensional.

Theorem 3.2.21

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Then all the following hold:

(a) U is finite-dimensional;

(b) $\dim(U) \leq \dim(V)$;

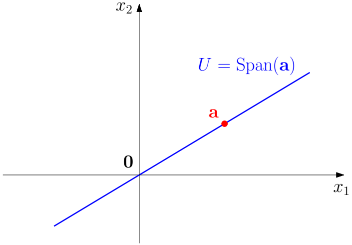
(c) if $\dim(U) = \dim(V)$, then $U = V$.

- **Warning:** Theorem 3.2.21(c) fails if V is infinite-dimensional!
  - Infinite-dimensional vector spaces can have proper subspaces that are infinite-dimensional.
  - For example, $\{p(x) \in \mathbb{P}_{\mathbb{R}} \mid p(0) = 0\}$ is an infinite-dimensional proper subspace of $\mathbb{P}_{\mathbb{R}}$.

- Let us consider a geometric interpretation of subspaces in $\mathbb{R}^n$.

- Let us consider a geometric interpretation of subspaces in $\mathbb{R}^n$.
- The only 0-dimensional subspace of $\mathbb{R}^n$ is $\{\mathbf{0}\}$.
    - This holds for any vector space V (not just $\mathbb{R}^n$), as long as the zero vector is from the vector space V in question. Recall that we defined $\text{Span}(\emptyset) = \{\mathbf{0}\}$, and obviously, $\emptyset$ is linearly independent.

- 1-dimensional subspaces of $\mathbb{R}^n$ are lines though the origin. Indeed, suppose that $\{\mathbf{a}\}$ is a basis of a subspace U of $\mathbb{R}^n$. Then $\mathbf{a} \neq \mathbf{0}$ (by linear independence), and we see that $U = \text{Span}(\mathbf{a})$ is the line through the origin and $\mathbf{a}$.
  - This is illustrated below for the case of $\mathbb{R}^2$.



So, 1-dimensional subspaces of $\mathbb{R}^n$ essentially look like copies of $\mathbb{R}^1$ inside of $\mathbb{R}^n$.

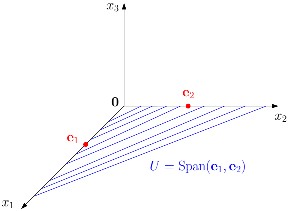
- 2-dimensional subspaces of $\mathbb{R}^n$ are planes through the origin.

- 2-dimensional subspaces of $\mathbb{R}^n$ are planes through the origin.
  - Indeed, suppose that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis of a subspace U of $\mathbb{R}^n$.

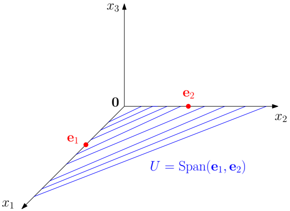
- 2-dimensional subspaces of $\mathbb{R}^n$ are planes through the origin.
  - Indeed, suppose that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis of a subspace U of $\mathbb{R}^n$.
  - By linear independence, $\mathbf{a}_1, \mathbf{a}_2$ are both non-zero and are not scalar multiples of each other.

- 2-dimensional subspaces of $\mathbb{R}^n$ are planes through the origin.
  - Indeed, suppose that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis of a subspace U of $\mathbb{R}^n$.
  - By linear independence, $\mathbf{a}_1, \mathbf{a}_2$ are both non-zero and are not scalar multiples of each other.
  - So, $U = \text{Span}(\mathbf{a}_1, \mathbf{a}_2)$ is the plane through the origin and through $\mathbf{a}_1$ and $\mathbf{a}_2$.

- 2-dimensional subspaces of $\mathbb{R}^n$ are planes through the origin.
  - Indeed, suppose that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis of a subspace U of $\mathbb{R}^n$.
  - By linear independence, $\mathbf{a}_1, \mathbf{a}_2$ are both non-zero and are not scalar multiples of each other.
  - So, $U = \text{Span}(\mathbf{a}_1, \mathbf{a}_2)$ is the plane through the origin and through $\mathbf{a}_1$ and $\mathbf{a}_2$.
  - For example, the subspace of $\mathbb{R}^3$ whose basis is $\{\mathbf{e}_1, \mathbf{e}_2\}$ is simply the x_1x_2-plane in $\mathbb{R}^3$ (illustrated below).



- 2-dimensional subspaces of $\mathbb{R}^n$ are planes through the origin.
  - Indeed, suppose that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis of a subspace U of $\mathbb{R}^n$.
  - By linear independence, $\mathbf{a}_1, \mathbf{a}_2$ are both non-zero and are not scalar multiples of each other.
  - So, $U = \text{Span}(\mathbf{a}_1, \mathbf{a}_2)$ is the plane through the origin and through $\mathbf{a}_1$ and $\mathbf{a}_2$.
  - For example, the subspace of $\mathbb{R}^3$ whose basis is $\{\mathbf{e}_1, \mathbf{e}_2\}$ is simply the x_1x_2-plane in $\mathbb{R}^3$ (illustrated below).



  - In general, 2-dimensional subspaces of $\mathbb{R}^n$ look like copies of $\mathbb{R}^2$ inside of $\mathbb{R}^n$ (of course, those copies of $\mathbb{R}^2$, i.e. planes, may possibly be "tilted," i.e. not formed by any two of the coordinate axes of $\mathbb{R}^n$); however, they must all pass through the origin.

- In general, for a positive integer $m \leq n$, an m-dimensional subspace of $\mathbb{R}^n$ looks like a copy of $\mathbb{R}^m$ inside of $\mathbb{R}^n$.

- In general, for a positive integer $m \leq n$, an m-dimensional subspace of $\mathbb{R}^n$ looks like a copy of $\mathbb{R}^m$ inside of $\mathbb{R}^n$.
  - Again, our copy of $\mathbb{R}^m$ may possibly be "tilted," i.e. not be formed by any m of the n axes of $\mathbb{R}^n$.
  - However, it must pass through the origin.

- Recall that if U and W are vector spaces over a field $\mathbb{F}$, then $U \times W$ is also a vector space over $\mathbb{F}$, with vector addition and scalar multiplication defined in a natural way, as follows:
  - $(\mathbf{u}_1, \mathbf{w}_1) + (\mathbf{u}_2, \mathbf{w}_2) := (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$;
  - $\alpha(\mathbf{u}, \mathbf{w}) := (\alpha\mathbf{u}, \alpha\mathbf{w})$ for all $\alpha \in \mathbb{F}$, $\mathbf{u} \in U$, and $\mathbf{w} \in W$.

- Recall that if U and W are vector spaces over a field $\mathbb{F}$, then $U \times W$ is also a vector space over $\mathbb{F}$, with vector addition and scalar multiplication defined in a natural way, as follows:
  - $(\mathbf{u}_1, \mathbf{w}_1) + (\mathbf{u}_2, \mathbf{w}_2) := (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$;
  - $\alpha(\mathbf{u}, \mathbf{w}) := (\alpha\mathbf{u}, \alpha\mathbf{w})$ for all $\alpha \in \mathbb{F}$, $\mathbf{u} \in U$, and $\mathbf{w} \in W$.
- We then have the following proposition:

Proposition 3.2.22

Let U and W be finite-dimensional vector spaces over a field $\mathbb{F}$. Then the vector space $U \times W$ is finite-dimensional, and moreover,

$$\dim(U \times W) = \dim(U) + \dim(W).$$

Proposition 3.2.22

Let U and W be finite-dimensional vector spaces over a field $\mathbb{F}$. Then the vector space $U \times W$ is finite-dimensional, and moreover,

$$\dim(U \times W) \;=\; \dim(U) + \dim(W).$$

Proof (outline).

Proposition 3.2.22

Let U and W be finite-dimensional vector spaces over a field $\mathbb{F}$. Then the vector space $U \times W$ is finite-dimensional, and moreover,

$$\dim(U \times W) \quad = \quad \dim(U) + \dim(W).$$

Proof (outline). Let $\mathbf{0}_U$ be the zero vector of the vector space U, and let $\mathbf{0}_W$ be the zero of the vector space W.

Proposition 3.2.22

Let U and W be finite-dimensional vector spaces over a field $\mathbb{F}$. Then the vector space $U \times W$ is finite-dimensional, and moreover,

$$\dim(U \times W) \quad = \quad \dim(U) + \dim(W).$$

Proof (outline). Let $\mathbf{0}_U$ be the zero vector of the vector space U, and let $\mathbf{0}_W$ be the zero of the vector space W. Set $m := \dim(U)$ and $n := \dim(W)$, and fix a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of U and a basis $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ of W.

Proposition 3.2.22

Let U and W be finite-dimensional vector spaces over a field $\mathbb{F}$. Then the vector space $U \times W$ is finite-dimensional, and moreover,

$$\dim(U \times W) \; = \; \dim(U) + \dim(W).$$

Proof (outline). Let $\mathbf{0}_U$ be the zero vector of the vector space U, and let $\mathbf{0}_W$ be the zero of the vector space W. Set $m := \dim(U)$ and $n := \dim(W)$, and fix a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of U and a basis $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ of W. It is then straightforward to check that

$$\Big\{(\mathbf{u}_1, \mathbf{0}_W), \dots, (\mathbf{u}_m, \mathbf{0}_W), (\mathbf{0}_U, \mathbf{w}_1), \dots, (\mathbf{0}_U, \mathbf{w}_n)\Big\}$$

is a basis of $U \times W$ (the details are left as an exercise),

Proposition 3.2.22

Let U and W be finite-dimensional vector spaces over a field $\mathbb{F}$. Then the vector space $U \times W$ is finite-dimensional, and moreover,

$$\dim(U \times W) \;\; = \;\; \dim(U) + \dim(W).$$

Proof (outline). Let $\mathbf{0}_U$ be the zero vector of the vector space U, and let $\mathbf{0}_W$ be the zero of the vector space W. Set $m := \dim(U)$ and $n := \dim(W)$, and fix a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of U and a basis $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ of W. It is then straightforward to check that

$$\Big\{(\mathbf{u}_1, \mathbf{0}_W), \dots, (\mathbf{u}_m, \mathbf{0}_W), (\mathbf{0}_U, \mathbf{w}_1), \dots, (\mathbf{0}_U, \mathbf{w}_n)\Big\}$$

is a basis of $U \times W$ (the details are left as an exercise), and consequently, $\dim(U \times W) = m + n = \dim(U) + \dim(W)$. $\square$

- Recall that if V is a vector space over a field $\mathbb{F}$, and U and W are subspaces of V, then $U \cap W$ and $U + W := \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \ \mathbf{w} \in W\}$ are also subspaces of V.

- Recall that if V is a vector space over a field $\mathbb{F}$, and U and W are subspaces of V, then $U \cap W$ and $U + W := \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \ \mathbf{w} \in W\}$ are also subspaces of V.

Theorem 3.2.23

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Then $U \cap W$ and $U + W$ are also finite-dimensional subspaces of V. Moreover, U, W, $U \cap W$, and $U + W$ are all finite-dimensional and satisfy

$$\dim(U + W) + \dim(U \cap W) \quad = \quad \dim(U) + \dim(W).$$

Proof (outline).

- Recall that if V is a vector space over a field $\mathbb{F}$, and U and W are subspaces of V, then $U \cap W$ and $U + W := \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \ \mathbf{w} \in W\}$ are also subspaces of V.

Theorem 3.2.23

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Then $U \cap W$ and $U + W$ are also finite-dimensional subspaces of V. Moreover, U, W, $U \cap W$, and $U + W$ are all finite-dimensional and satisfy

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

Proof (outline). The proof of the fact that $U \cap W$ and $U + W$ are subspaces of V was an exercise.

- Recall that if V is a vector space over a field $\mathbb{F}$, and U and W are subspaces of V, then $U \cap W$ and $U + W := \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \ \mathbf{w} \in W\}$ are also subspaces of V.

Theorem 3.2.23

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Then $U \cap W$ and $U + W$ are also finite-dimensional subspaces of V. Moreover, U, W, $U \cap W$, and $U + W$ are all finite-dimensional and satisfy

$$\dim(U + W) + \dim(U \cap W) \ = \ \dim(U) + \dim(W).$$

Proof (outline). The proof of the fact that $U \cap W$ and $U + W$ are subspaces of V was an exercise. Since V is finite-dimensional, Theorem 3.2.21 guarantees that all its subspaces are finite dimensional; in particular, U, W, $U \cap W$, and $U + W$ are all finite-dimensional.

Theorem 3.2.23

$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$.

Proof (continued). Set $m := \dim(U)$, $n := \dim(W)$, and $p := \dim(U \cap W)$.

Theorem 3.2.23

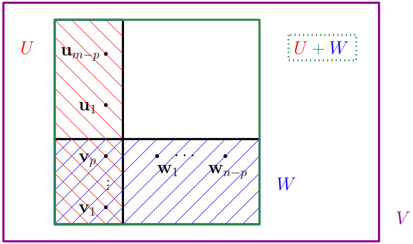
$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$.

Proof (continued). Set $m := \dim(U)$, $n := \dim(W)$, and $p := \dim(U \cap W)$. Fix a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of $U \cap W$.

Theorem 3.2.23

$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$.

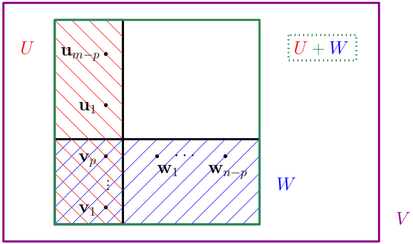
Proof (continued). Set $m := \dim(U)$, $n := \dim(W)$, and $p := \dim(U \cap W)$. Fix a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of $U \cap W$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly independent set in the finite-dimensional vector space U, and so by Theorem 3.2.19, it can be extended to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{u}_1, \ldots, \mathbf{u}_{m-p}\}$ of U. Similarly, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ can be extended to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_{n-p}\}$ of W.



Theorem 3.2.23

$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$.

Proof (continued).



It is now straightforward to check that

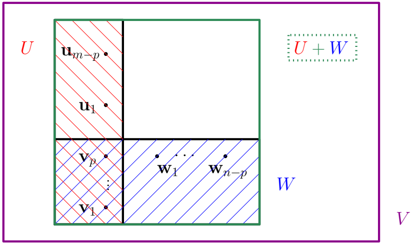
$$\left\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{u}_1, \ldots, \mathbf{u}_{m-p}, \mathbf{w}_1, \ldots, \mathbf{w}_{n-p}\right\}$$

is a basis of $U + W$ (details: exercise).

Theorem 3.2.23

$\dim(U+W) + \dim(U \cap W) = \dim(U) + \dim(W)$.

Proof (continued).



It is now straightforward to check that

$$\left\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{u}_1, \ldots, \mathbf{u}_{m-p}, \mathbf{w}_1, \ldots, \mathbf{w}_{n-p}\right\}$$

is a basis of $U+W$ (details: exercise). So,

$$\dim(U+W) \quad = \quad p + (m-p) + (n-p) \quad = \quad m+n-p.$$

Theorem 3.2.23

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Then $U \cap W$ and $U + W$ are also finite-dimensional subspaces of V. Moreover, U, W, $U \cap W$, and $U + W$ are all finite-dimensional and satisfy

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

Proof (continued). It now follows that

$$\begin{aligned} \dim(U + W) + \dim(U \cap W) &= (m + n - p) + p \\ &= m + n \\ &= \dim(U) + \dim(W), \end{aligned}$$

which is what we needed to show. □

Theorem 3.2.23

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Then $U \cap W$ and $U + W$ are also finite-dimensional subspaces of V. Moreover, U, W, $U \cap W$, and $U + W$ are all finite-dimensional and satisfy

$$\dim(U + W) + \dim(U \cap W) \;=\; \dim(U) + \dim(W).$$

Theorem 3.2.23

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Then $U \cap W$ and $U + W$ are also finite-dimensional subspaces of V. Moreover, U, W, $U \cap W$, and $U + W$ are all finite-dimensional and satisfy

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

- If V is a vector space over a field $\mathbb{F}$ and U and W are its subspaces s.t. $U \cap W = \{\mathbf{0}\}$ and $V = U + W$, then we say that V is the *direct sum* of U and W, and we write $V = U \oplus W$.

Theorem 3.2.23

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Then $U \cap W$ and $U + W$ are also finite-dimensional subspaces of V. Moreover, U, W, $U \cap W$, and $U + W$ are all finite-dimensional and satisfy

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

- If V is a vector space over a field $\mathbb{F}$ and U and W are its subspaces s.t. $U \cap W = \{\mathbf{0}\}$ and $V = U + W$, then we say that V is the *direct sum* of U and W, and we write $V = U \oplus W$.
  - If $V = U \oplus W$ is also finite-dimensional, then Theorem 3.2.23 immediately implies that $\dim(V) = \dim(U) + \dim(W)$.

Theorem 3.2.23

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Then $U \cap W$ and $U + W$ are also finite-dimensional subspaces of V. Moreover, U, W, $U \cap W$, and $U + W$ are all finite-dimensional and satisfy

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

- If V is a vector space over a field $\mathbb{F}$ and U and W are its subspaces s.t. $U \cap W = \{\mathbf{0}\}$ and $V = U + W$, then we say that V is the *direct sum* of U and W, and we write $V = U \oplus W$.
  - If $V = U \oplus W$ is also finite-dimensional, then Theorem 3.2.23 immediately implies that $\dim(V) = \dim(U) + \dim(W)$.
  - This is because $\dim(U \cap W) = 0$.

Theorem 3.2.23

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Then $U \cap W$ and $U + W$ are also finite-dimensional subspaces of V. Moreover, U, W, $U \cap W$, and $U + W$ are all finite-dimensional and satisfy

$$\dim(U + W) + \dim(U \cap W) \quad = \quad \dim(U) + \dim(W).$$

- If V is a vector space over a field $\mathbb{F}$ and U and W are its subspaces s.t. $U \cap W = \{\mathbf{0}\}$ and $V = U + W$, then we say that V is the *direct sum* of U and W, and we write $V = U \oplus W$.
  - If $V = U \oplus W$ is also finite-dimensional, then Theorem 3.2.23 immediately implies that $\dim(V) = \dim(U) + \dim(W)$.
  - This is because $\dim(U \cap W) = 0$.
- Moreover, we have the following theorem (next slide).

Theorem 3.2.24

Let V be a vector space over a field $\mathbb{F}$, and let U and W be subspaces of V s.t. $V = U \oplus W$. Then for all $\mathbf{v} \in V$, there exist unique $\mathbf{u} \in U$ and $\mathbf{w} \in W$ s.t. $\mathbf{v} = \mathbf{u} + \mathbf{w}$.

Proof. Exercise.

- Optional reading: subsection 3.2.7 from the Lecture Notes ("A very brief introduction to infinite bases").