Linear Algebra 1

Lecture #4

Invertible matrices

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Definition

A **square** matrix $A \in \mathbb{F}^{n \times n}$ (where $\mathbb{F}$ is a field) is *invertible* if there exists a matrix $B \in \mathbb{F}^{n \times n}$, called an *inverse* of A, s.t. $AB = BA = I_n$. A square matrix that is not invertible is called *non-invertible*.

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Example 1.11.2

The matrix $A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in $\mathbb{R}^{2\times 2}$ is invertible, and its inverse is $A^{-1} := \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, which we can easily verify by checking that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = I_2 \quad \text{and} \quad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = I_2.$$

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- We will soon give a recipe for checking whether a matrix is invertible, and if so, for finding its inverse.
- But first, we discuss some alternative terminology (i.e. other terms for invertible matrices), and we give a proof of Proposition 1.11.1.

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  - The Czech term for an invertible matrix is "regulární matice," and for this reason, Czech mathematicians sometimes use the term "regular matrix" instead of "invertible matrix"; however, this usage ("regular matrix") is quite rare in the English speaking world.

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  - The Czech term for an invertible matrix is “regulární matice,” and for this reason, Czech mathematicians sometimes use the term “regular matrix” instead of “invertible matrix”; however, this usage (“regular matrix”) is quite rare in the English speaking world.
  - In this course, we will consistently use the term “invertible matrix.”

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be an invertible matrix. Then A has a unique inverse.

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$$\begin{array}{rcll} B & = & BI_n & \text{by Proposition 1.7.2} \\ & = & B(AC) & \text{because } AC = I_n \\ & = & (BA)C & \begin{array}{l}\text{by the associativity of} \\ \text{matrix multiplication}\end{array} \\ & = & I_nC & \text{because } BA = I_n \\ & = & C & \text{by Proposition 1.7.2.} \end{array}$$

This completes the argument. □

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Proposition 1.11.3

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Assume that A is invertible and that $AB = I_n$ or $BA = I_n$. Then $A^{-1} = B$.

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  - For the case when $BA = I_n$, this is virtually identical to the proof of Proposition 1.11.1 (details: Lecture Notes).
  - The case when $AB = I_n$ is similar (details: exercise).

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- **Remark:** Note that Proposition 1.11.3 can only be applied if we already know that A is invertible.
    - Once we have developed a lot more theory, we will be able to eliminate this hypothesis and show that if $A, B \in \mathbb{F}^{n \times n}$ are **square** matrices that satisfy $AB = I_n$, then both A and B are invertible and are each other's inverses.

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  - Once we have developed a lot more theory, we will be able to eliminate this hypothesis and show that if $A, B \in \mathbb{F}^{n \times n}$ are **square** matrices that satisfy $AB = I_n$, then both A and B are invertible and are each other's inverses.
  - However, we cannot prove this stronger statement yet, and therefore, we cannot use it yet.

Theorem 1.11.4

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$ be a square matrix, and set $\left[\begin{array}{c:c} U & B \end{array}\right] = \mathrm{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right])$, where each of U and B has n columns. Then

- ⓐ if $U = I_n$, then A is invertible and $B = A^{-1}$;
- ⓑ if $U \neq I_n$, then A is not invertible.

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- (a) if $U = I_n$, then A is invertible and $B = A^{-1}$;
- (b) if $U \neq I_n$, then A is not invertible.

- Theorem 1.11.4 gives a recipe for checking if a square matrix is invertible, and if so, for finding its inverse.
- We first consider an example, and then we develop the theory that we need to actually prove Theorem 1.11.4.

Example 1.11.5

Consider the following matrices.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, with entries understood to be in $\mathbb{R}$;
2. $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, with entries understood to be in $\mathbb{Z}_2$;
3. $C = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, with entries understood to be in $\mathbb{Z}_3$.

For each of these three matrices, determine if the matrix is invertible, and if so, find its inverse.

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(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, with entries understood to be in $\mathbb{R}$.

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Solution. We form the matrix

$$\left[\begin{array}{c|c} A & I_2 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right],$$

and by row reducing, we obtain

$$\mathrm{RREF}\left(\left[\begin{array}{c|c} A & I_2 \end{array}\right]\right) = \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right].$$

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The submatrix of $\mathrm{RREF}\left(\left[\begin{array}{c:c} A & I_2 \end{array}\right]\right)$ to the left of the vertical dotted line is I_2. So, A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

$\square$

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(b) $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, with entries understood to be in $\mathbb{Z}_2$.

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ⓑ $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, with entries understood to be in $\mathbb{Z}_2$.

Solution. We form the matrix

$$\left[\begin{array}{c:c} B & I_3 \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right],$$

and by row reducing, we obtain

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ⓑ $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, with entries understood to be in $\mathbb{Z}_2$.

Solution (continued). Reminder:

$$\mathrm{RREF}\Big(\left[\begin{array}{c:c} B & I_3 \end{array}\right]\Big) = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

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The submatrix of $\operatorname{RREF}(\left[\begin{array}{c:c} B & I_3 \end{array}\right])$ to the left of the vertical dotted line is I_3. So, B is invertible, and its inverse is

$$B^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

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$$\left[\begin{array}{c:c} C & I_3 \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right],$$

and by row reducing, we obtain

$$\mathrm{RREF}\left(\left[\begin{array}{c:c} C & I_3 \end{array}\right]\right) = \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array}\right].$$

The submatrix of $\mathrm{RREF}\left(\left[\begin{array}{c:c} C & I_3 \end{array}\right]\right)$ to the left of the vertical dotted line is not I_3. So, C is **not** invertible. □

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- The main goals for the remainder of this lecture are:
  - to prove Theorem 1.11.4;
    - First, we prove some basic results about invertible matrices.
    - Then, we introduce "elementary matrices" (matrices obtained by applying an elementary row operation to the identity matrix I_n), and we prove some results about such matrices.
    - Finally, using all this, we prove Theorem 1.11.4.

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        - Finally, using all this, we prove Theorem 1.11.4.
    - to state and prove the first version of the Invertible Matrix Theorem (which gives a long list of statements about a square matrix A that are equivalent to A being invertible).

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Theorem 1.11.6

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- Theorem 1.11.6 is one of the main reasons we care about invertible matrices.
  - Note that it implies that if the **coefficient** matrix of a linear system is invertible, then that linear system has a unique solution.
- We first take a look at an example, and then we prove Theorem 1.11.6.

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Example 1.11.7

Set

$$A \ := \ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} \ := \ \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.

Solution.

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Solution. As we saw in Example 1.11.2, the matrix A is invertible, and its inverse is

$$A^{-1} \;=\; \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

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Solution (continued). So, by Theorem 1.11.6, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} \quad = \quad A^{-1}\mathbf{b} \quad = \quad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad = \quad \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

□

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Proof. Fix any vector $\mathbf{b} \in \mathbb{F}^n$. To show that $A^{-1}\mathbf{b}$ is indeed a solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, we compute

$$A(A^{-1}\mathbf{b}) \overset{(*)}{=} (\underbrace{AA^{-1}}_{=I_n})\mathbf{b} = I_n\mathbf{b} \overset{(**)}{=} \mathbf{b},$$

where (*) follows from Corollary 1.7.6(g), and (**) follows from Proposition 1.4.5.

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So far, we have proven that $A^{-1}\mathbf{b}$ is a solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.

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So far, we have proven that $A^{-1}\mathbf{b}$ is a solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$. It remains to prove uniqueness (next slide).

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Proof (continued). Fix any solution $\mathbf{x}_0 \in \mathbb{F}^n$ of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.

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Proof (continued). Fix any solution $\mathbf{x}_0 \in \mathbb{F}^n$ of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{x}_0 = \mathbf{b}$, and consequently, $A^{-1}(A\mathbf{x}_0) = A^{-1}\mathbf{b}$.

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Proof (continued). Fix any solution $\mathbf{x}_0 \in \mathbb{F}^n$ of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{x}_0 = \mathbf{b}$, and consequently, $A^{-1}(A\mathbf{x}_0) = A^{-1}\mathbf{b}$. We now compute:

$$A^{-1}\mathbf{b} \;=\; A^{-1}(A\mathbf{x}_0) \;\overset{(*)}{=}\; (\underbrace{A^{-1}A}_{=I_n})\mathbf{x}_0 \;=\; I_n\mathbf{x}_0 \;\overset{(**)}{=}\; \mathbf{x}_0.$$

where once again, (*) follows from Corollary 1.7.6(g), and (**) follows from Proposition 1.4.5. This proves that $A^{-1}\mathbf{b}$ is in fact the unique solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$. ☐

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- We already saw how one can check if a square matrix (with entries in some field) is invertible, and if so, how one can compute its inverse.
    - Granted, we still need to prove that this “recipe” (Theorem 1.11.4) works.

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- However, if we do not already know whether A is invertible (or we know that A is invertible, but have not yet computed its inverse), then the most efficient way to solve our matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is by row reducing the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$.

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- However, if we do not already know whether A is invertible (or we know that A is invertible, but have not yet computed its inverse), then the most efficient way to solve our matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is by row reducing the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$.
- Using the formula $\mathbf{x} = A^{-1}\mathbf{b}$ is only efficient if we already happen to know that A is invertible and have already computed its inverse A^{-1} for some reason other than solving the equation $A\mathbf{x} = \mathbf{b}$.

Proposition 1.11.8

Let $\mathbb{F}$ be a field. Then all the following hold:

- (a) the identity matrix I_n is invertible and is its own inverse (i.e. $I_n^{-1} = I_n$);
- (b) if a matrix $A \in \mathbb{F}^{n \times n}$ is invertible, then its inverse A^{-1} is also invertible, and moreover, $(A^{-1})^{-1} = A$;
- (c) if a matrix $A \in \mathbb{F}^{n \times n}$ is invertible, then its transpose A^T is also invertible, and moreover, $(A^T)^{-1} = (A^{-1})^T$;
- (d) if matrices $A, B \in \mathbb{F}^{n \times n}$ are invertible matrices, then AB is also invertible, and moreover, $(AB)^{-1} = B^{-1}A^{-1}$;
- (e) if matrices $A_1, \dots, A_k \in \mathbb{F}^{n \times n}$ are invertible, then the matrix $A_1 \dots A_k$ is also invertible, and moreover, $(A_1 \dots A_k)^{-1} = A_k^{-1} \dots A_1^{-1}$;
- (f) if a matrix $A \in \mathbb{F}^{n \times n}$ is invertible, then for all non-negative integers m, the matrix A^m is also invertible, and moreover, $(A^m)^{-1} = (A^{-1})^m$.

Definition

A **square** matrix $A \in \mathbb{F}^{n\times n}$ (where $\mathbb{F}$ is a field) is *invertible* if there exists a matrix $B \in \mathbb{F}^{n\times n}$, called an *inverse* of A, s.t. $AB = BA = I_n$. A square matrix that is not invertible is called *non-invertible*.

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- (a) the identity matrix I_n is invertible and is its own inverse (i.e. $I_n^{-1} = I_n$)

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where (*) follows from Proposition 1.8.1(d). An analogous argument shows that $(A^{-1})^T A^T = I_n$.

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Proof. Fix invertible matrices $A, B \in \mathbb{F}^{n\times n}$. It suffices to show that $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n$. For this, we compute (using the associativity of matrix multiplication):

- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$;
- $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$.

$\square$

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Proof. Part (e) follows from (d) via an easy induction on k (the details are left as an exercise). $\square$

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- (f) if a matrix $A \in \mathbb{F}^{n \times n}$ is invertible, then for all non-negative integers m, the matrix A^m is also invertible, and moreover, $(A^m)^{-1} = (A^{-1})^m$.

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- Note that this is only defined if A is invertible (and is undefined otherwise).

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(a) the identity matrix I_n is invertible and is its own inverse (i.e. $I_n^{-1} = I_n$);

(b) if a matrix $A \in \mathbb{F}^{n \times n}$ is invertible, then its inverse A^{-1} is also invertible, and moreover, $(A^{-1})^{-1} = A$;

(c) if a matrix $A \in \mathbb{F}^{n \times n}$ is invertible, then its transpose A^T is also invertible, and moreover, $(A^T)^{-1} = (A^{-1})^T$;

(d) if matrices $A, B \in \mathbb{F}^{n \times n}$ are invertible matrices, then AB is also invertible, and moreover, $(AB)^{-1} = B^{-1}A^{-1}$;

(e) if matrices $A_1, \dots, A_k \in \mathbb{F}^{n \times n}$ are invertible, then the matrix $A_1 \dots A_k$ is also invertible, and moreover, $(A_1 \dots A_k)^{-1} = A_k^{-1} \dots A_1^{-1}$;

(f) if a matrix $A \in \mathbb{F}^{n \times n}$ is invertible, then for all non-negative integers m, the matrix A^m is also invertible, and moreover, $(A^m)^{-1} = (A^{-1})^m$.

Theorem 1.11.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$ be a square matrix, and let $f : \mathbb{F}^n \to \mathbb{F}^n$ be given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$. Then f is linear and its standard matrix is A. Furthermore, the following are equivalent:

- (a) f is an isomorphism;
- (b) A is invertible;
- (c) $\mathrm{RREF}(A) = I_n$;
- (d) $\mathrm{rank}(A) = n$.

Moreover, in this case, f^{-1} is an isomorphism and its standard matrix is A^{-1}.

Proof.

Theorem 1.11.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$ be a square matrix, and let $f : \mathbb{F}^n \to \mathbb{F}^n$ be given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$. Then f is linear and its standard matrix is A. Furthermore, the following are equivalent:

ⓐ f is an isomorphism;

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ⓓ $\text{rank}(A) = n$.

Moreover, in this case, f^{-1} is an isomorphism and its standard matrix is A^{-1}.

Proof. The function f is a matrix transformation, and so by Proposition 1.10.4, it is linear. The fact that A is its standard matrix follows from the definition of a standard matrix.

Proof (continued). By Theorem 1.10.19, (a) and (d) are equivalent, *Proof (continued).* By Theorem 1.10.19, (a) and (d) are equivalent, and by Proposition 1.6.7, (c) and (d) are equivalent.

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It now suffices to prove the following:

Proof (continued). By Theorem 1.10.19, (a) and (d) are equivalent, and by Proposition 1.6.7, (c) and (d) are equivalent. So, (a), (c), and (d) are equivalent. Moreover, Proposition 1.10.20 guarantees that if f is an isomorphism, then so is f^{-1}.

It now suffices to prove the following:

(1) if f is an isomorphism, then A is invertible, and moreover, the standard matrix of f^{-1} is A^{-1};

  - Note that (1) states that (a) implies (b), and moreover, that if (a) holds, then the standard matrix of f^{-1} is A^{-1}.

Proof (continued). By Theorem 1.10.19, (a) and (d) are equivalent, and by Proposition 1.6.7, (c) and (d) are equivalent. So, (a), (c), and (d) are equivalent. Moreover, Proposition 1.10.20 guarantees that if f is an isomorphism, then so is f^{-1}.

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(2) if A is invertible, then f is an isomorphism.

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This proves that $f \circ g = g \circ f = \mathrm{Id}_{\mathbb{F}^n}$, and it follows that f is indeed a bijection. $\square$

Theorem 1.11.9

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$ be a square matrix, and let $f : \mathbb{F}^n \to \mathbb{F}^n$ be given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$. Then f is linear and its standard matrix is A. Furthermore, the following are equivalent:

- ⓐ f is an isomorphism;
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- ⓒ $\text{RREF}(A) = I_n$;
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Moreover, in this case, f^{-1} is an isomorphism and its standard matrix is A^{-1}.

Corollary 1.11.10

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be a square matrix. Then the following are equivalent:

- (a) A is invertible;
- (b) A^T is invertible;
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Proof. By Theorem 1.11.9 applied to the matrix A, we have that (a) and (c) are equivalent. Similarly, by Theorem 1.11.10 applied to the matrix A^T, we have that (b) and (d) are equivalent.

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By Proposition 1.11.8(c) applied to the matrix A, we have that (a) implies (b).

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By Proposition 1.11.8(c) applied to the matrix A, we have that (a) implies (b). On the other hand, Proposition 1.11.8(c) applied to A^T guarantees that if A^T is invertible, then so is $(A^T)^T = A$, and so (b) implies (a). □

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- By Corollary 1.11.10, a square matrix (with entries in some field) has full rank iff its transpose has full rank.

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(d) $\text{rank}(A^T) = n$.

- By Corollary 1.11.10, a square matrix (with entries in some field) has full rank iff its transpose has full rank.
- In fact, the rank of any matrix is equal to the rank of its transpose, but we cannot prove this yet.

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- For an elementary row operation performed on a matrix with n rows, the elementary matrix that *corresponds* to this elementary row operation is the matrix obtained by performing that same elementary row operation on the identity matrix I_n.
- Let us consider some examples.

1. The elementary matrix that corresponds to swapping rows 2 and 4 ("$R_2 \leftrightarrow R_4$") of a matrix with 5 rows is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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2. The elementary matrix that corresponds to multiplying the second row of a matrix with three rows by a scalar $\alpha \neq 0$ (“$R_2 \to \alpha R_2$”) is
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3. The elementary matrix that corresponds to adding α times the third row to the second row (“$R_2 \to R_2 + \alpha R_3$”) of a matrix with three rows is
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 1.11.11

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ be a matrix. Then all the following hold:

(a) if R is any elementary row operation (performed on a matrix with n rows and with entries in $\mathbb{F}$) and E is the corresponding elementary matrix, then the matrix obtained from A by performing R on it is precisely the matrix EA;

(b) if $R_1, \dots, R_k$ are elementary row operations (performed on a matrix with n rows and with entries in $\mathbb{F}$) and $E_1, \dots, E_k \in \mathbb{F}^{n \times n}$ are, respectively, the corresponding elementary matrices, then the matrix obtained from A by performing $R_1, \dots, R_k$ (in that order) on it is precisely the matrix $E_k \dots E_1 A$.

- Part (b) follows from (a) via an easy induction (details: exercise).

$$A \overset{R_1}{\sim} E_1 A \overset{R_2}{\sim} E_2 E_1 A \overset{R_3}{\sim} E_3 E_2 E_1 A \overset{R_4}{\sim} \dots \overset{R_k}{\sim} E_k \dots E_3 E_2 E_1 A.$$

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(a) if R is any elementary row operation (performed on a matrix with n rows and with entries in $\mathbb{F}$) and E is the corresponding elementary matrix, then the matrix obtained from A by performing R on it is precisely the matrix EA;

Proof of (a).

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Proof of (a). Consider any elementary row operation R performed on a matrix with n rows (and with entries in the field $\mathbb{F}$).

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Proof of (a). Consider any elementary row operation R performed on a matrix with n rows (and with entries in the field $\mathbb{F}$). Define $f_R : \mathbb{F}^n \to \mathbb{F}^n$ by, for each $\mathbf{u} \in \mathbb{F}^n$, letting $f(\mathbf{u})$ be the vector obtained by performing the elementary row operation R on $\mathbf{u}$.

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$$\left[\begin{array}{ccc} f_R(\mathbf{e}_1) & \dots & f_R(\mathbf{e}_n) \end{array} \right],$$

which is precisely the matrix obtained from I_n by applying the elementary row operation R to it, and this matrix is precisely the elementary matrix E.

Proposition 1.11.11

(a) if R is any elementary row operation (performed on a matrix with n rows and with entries in $\mathbb{F}$) and E is the corresponding elementary matrix, then the matrix obtained from A by performing R on it is precisely the matrix EA;

Proof of (a). Reminder: $f_R : \mathbb{F}^n \to \mathbb{F}^n$ performs R on each vector in $\mathbb{F}^n$, it is linear, and its standard matrix is E.

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Now, fix any matrix $A \in \mathbb{F}^{n \times m}$, and set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$.

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Proof of (a). Reminder: $f_R : \mathbb{F}^n \to \mathbb{F}^n$ performs R on each vector in $\mathbb{F}^n$, it is linear, and its standard matrix is E.

Now, fix any matrix $A \in \mathbb{F}^{n \times m}$, and set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$. Then

$$EA = \begin{bmatrix} E\mathbf{a}_1 & \dots & E\mathbf{a}_m \end{bmatrix} \overset{(*)}{=} \begin{bmatrix} f_R(\mathbf{a}_1) & \dots & f_R(\mathbf{a}_m) \end{bmatrix} =: M,$$

where (*) follows from the fact that E is the standard matrix of f_R. But obviously, the matrix M is precisely the matrix obtained by performing the elementary row operation R on A. This proves (a). □

Proposition 1.11.11

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$ be a matrix. Then all the following hold:

- (a) if R is any elementary row operation (performed on a matrix with n rows and with entries in $\mathbb{F}$) and E is the corresponding elementary matrix, then the matrix obtained from A by performing R on it is precisely the matrix EA;
- (b) if $R_1, \dots, R_k$ are elementary row operations (performed on a matrix with n rows and with entries in $\mathbb{F}$) and $E_1, \dots, E_k \in \mathbb{F}^{n\times n}$ are, respectively, the corresponding elementary matrices, then the matrix obtained from A by performing $R_1, \dots, R_k$ (in that order) on it is precisely the matrix $E_k \dots E_1 A$.

Proposition 1.11.12

Let $\mathbb{F}$ be a field. Then all the following hold:

1. elementary matrices in $\mathbb{F}^{n \times n}$ are invertible;
2. the inverse of an elementary matrix in $\mathbb{F}^{n \times n}$ is an elementary matrix in $\mathbb{F}^{n \times n}$;
3. a matrix $A \in \mathbb{F}^{n \times n}$ is invertible iff there exist elementary matrices $E_1, \dots, E_k$ s.t. $A = E_1 \dots E_k$ (that is, a matrix is invertible iff it can be written as a product of elementary matrices).

Proof.

Proposition 1.11.12

Let $\mathbb{F}$ be a field. Then all the following hold:

- (a) elementary matrices in $\mathbb{F}^{n \times n}$ are invertible;
- (b) the inverse of an elementary matrix in $\mathbb{F}^{n \times n}$ is an elementary matrix in $\mathbb{F}^{n \times n}$;
- (c) a matrix $A \in \mathbb{F}^{n \times n}$ is invertible iff there exist elementary matrices $E_1, \dots, E_k$ s.t. $A = E_1 \dots E_k$ (that is, a matrix is invertible iff it can be written as a product of elementary matrices).

Proof. We prove (a) and (b) simultaneously, and we prove (c) separately.

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1. elementary matrices in $\mathbb{F}^{n\times n}$ are invertible;
2. the inverse of an elementary matrix in $\mathbb{F}^{n\times n}$ is an elementary matrix in $\mathbb{F}^{n\times n}$;

Proof of (a) and (b).

Proposition 1.11.12

- (a) elementary matrices in $\mathbb{F}^{n \times n}$ are invertible;
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Proof of (a) and (b). Let R be an elementary row operation performed on a matrix with n rows (and with entries in the field $\mathbb{F}$), and let E be the elementary matrix that corresponds to R.

Proposition 1.11.12

ⓐ elementary matrices in $\mathbb{F}^{n \times n}$ are invertible;

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Proof of (a) and (b). Let R be an elementary row operation performed on a matrix with n rows (and with entries in the field $\mathbb{F}$), and let E be the elementary matrix that corresponds to R. Let R' be the elementary row operation that "undoes" R, and let E' be the elementary matrix that corresponds to R'.

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- Essentially (and slightly informally):

$$I_n \quad \overset{R}{\sim} \quad EI_n \quad \overset{R'}{\sim} \quad \underbrace{E'EI_n}_{=E'E} \quad \overset{\substack{\text{because } R' \\ \text{undoes } R}}{=} \quad I_n$$

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This proves that E is invertible, and that its inverse is the elementary matrix E'. This proves (a) and (b).

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Let $\mathbb{F}$ be a field. Then all the following hold:

- (a) elementary matrices in $\mathbb{F}^{n \times n}$ are invertible;
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- (c) a matrix $A \in \mathbb{F}^{n \times n}$ is invertible iff there exist elementary matrices $E_1, \dots, E_k$ s.t. $A = E_1 \dots E_k$ (that is, a matrix is invertible iff it can be written as a product of elementary matrices).

Proof of (c).

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Let $\mathbb{F}$ be a field. Then all the following hold:

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Proof of (c). The fact that products of elementary matrices are invertible follows immediately from part (a) and from the fact that (by Proposition 1.11.8(e)) products of invertible matrices are invertible.

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Let $\mathbb{F}$ be a field. Then all the following hold:

- (a) elementary matrices in $\mathbb{F}^{n\times n}$ are invertible;
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- (c) a matrix $A \in \mathbb{F}^{n\times n}$ is invertible iff there exist elementary matrices $E_1, \dots, E_k$ s.t. $A = E_1 \dots E_k$ (that is, a matrix is invertible iff it can be written as a product of elementary matrices).

Proof of (c). The fact that products of elementary matrices are invertible follows immediately from part (a) and from the fact that (by Proposition 1.11.8(e)) products of invertible matrices are invertible.

For the reverse direction, we fix an arbitrary invertible matrix $A \in \mathbb{F}^{n\times n}$, and we show that A can be written as a product of elementary matrices.

Proposition 1.11.12

- (c) a matrix $A \in \mathbb{F}^{n \times n}$ is invertible iff there exist elementary matrices $E_1, \dots, E_k$ s.t. $A = E_1 \dots E_k$ (that is, a matrix is invertible iff it can be written as a product of elementary matrices).

Proof of (c) (continued). Reminder: $A \in \mathbb{F}^{n \times n}$ is invertible; WTS A is a product of elementary matrices.

Proposition 1.11.12

- (c) a matrix $A \in \mathbb{F}^{n\times n}$ is invertible iff there exist elementary matrices $E_1, \dots, E_k$ s.t. $A = E_1 \dots E_k$ (that is, a matrix is invertible iff it can be written as a product of elementary matrices).

Proof of (c) (continued). Reminder: $A \in \mathbb{F}^{n\times n}$ is invertible; WTS A is a product of elementary matrices.

Since A is invertible, Proposition 1.11.9 guarantees that $\mathrm{RREF}(A) = I_n$.

Proposition 1.11.12

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Proof of (c) (continued). Reminder: $A \in \mathbb{F}^{n \times n}$ is invertible; WTS A is a product of elementary matrices.

Since A is invertible, Proposition 1.11.9 guarantees that $\text{RREF}(A) = I_n$. In particular, A and I_n are row equivalent, and it follows that we can transform I_n into A via some sequence $R_1, \dots, R_k$ of elementary row operations:

$$I_n \overset{R_1}{\sim} \dots \overset{R_k}{\sim} A.$$

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- (c) a matrix $A \in \mathbb{F}^{n\times n}$ is invertible iff there exist elementary matrices $E_1, \dots, E_k$ s.t. $A = E_1 \dots E_k$ (that is, a matrix is invertible iff it can be written as a product of elementary matrices).

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$$I_n \overset{R_1}{\sim} \dots \overset{R_k}{\sim} A.$$

For each index $i \in \{1, \dots, k\}$, let $E_i \in \mathbb{F}^{n\times n}$ be the elementary matrix that corresponds to the elementary row operation R_i.

Proposition 1.11.12

- (c) a matrix $A \in \mathbb{F}^{n\times n}$ is invertible iff there exist elementary matrices $E_1, \dots, E_k$ s.t. $A = E_1 \dots E_k$ (that is, a matrix is invertible iff it can be written as a product of elementary matrices).

Proof of (c) (continued). Reminder: $A \in \mathbb{F}^{n\times n}$ is invertible; WTS A is a product of elementary matrices.

Since A is invertible, Proposition 1.11.9 guarantees that $\text{RREF}(A) = I_n$. In particular, A and I_n are row equivalent, and it follows that we can transform I_n into A via some sequence $R_1, \dots, R_k$ of elementary row operations:

$$I_n \overset{R_1}{\sim} \dots \overset{R_k}{\sim} A.$$

For each index $i \in \{1, \dots, k\}$, let $E_i \in \mathbb{F}^{n\times n}$ be the elementary matrix that corresponds to the elementary row operation R_i. But then by Proposition 1.11.11(b), we have that $A = E_k \dots E_1 I_n = E_k \dots E_1$. This proves (c). $\square$

Proposition 1.11.12

Let $\mathbb{F}$ be a field. Then all the following hold:

- (a) elementary matrices in $\mathbb{F}^{n\times n}$ are invertible;
- (b) the inverse of an elementary matrix in $\mathbb{F}^{n\times n}$ is an elementary matrix in $\mathbb{F}^{n\times n}$;
- (c) a matrix $A \in \mathbb{F}^{n\times n}$ is invertible iff there exist elementary matrices $E_1, \dots, E_k$ s.t. $A = E_1 \dots E_k$ (that is, a matrix is invertible iff it can be written as a product of elementary matrices).

Theorem 1.11.13

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times m}$. Then the following are equivalent:

- ⓐ $A \sim B$;
- ⓑ there exist elementary matrices $E_1, \dots, E_k \in \mathbb{F}^{n \times n}$ s.t. $B = E_1 \dots E_k A$;
- ⓒ there exists an invertible matrix $C \in \mathbb{F}^{n \times n}$ s.t. $B = CA$.

Proof.

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Proof. By definition, (a) is equivalent to:

(a') B can be obtained from A via some sequence of elementary row operations.

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Proof. By definition, (a) is equivalent to:

(a') B can be obtained from A via some sequence of elementary row operations.

But Proposition 1.11.11(b) guarantees that (a') and (b) are equivalent, and Proposition 1.11.12(c) guarantees that (b) and (c) are equivalent. This completes the argument. □

Theorem 1.11.4

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$ be a square matrix, and set $\left[\begin{array}{c:c} U & B \end{array}\right] = \text{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right])$, where each of U and B has n columns. Then

- (a) if $U = I_n$, then A is invertible and $B = A^{-1}$;
- (b) if $U \neq I_n$, then A is not invertible.

Proof.

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Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$ be a square matrix, and set $\left[\begin{array}{c:c} U & B \end{array}\right] = \mathrm{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right])$, where each of U and B has n columns. Then

- ⓐ if $U = I_n$, then A is invertible and $B = A^{-1}$;
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Proof. By Theorem 1.11.9, we have that A is invertible iff $\mathrm{RREF}(A) = I_n$,

Theorem 1.11.4

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- (a) if $U = I_n$, then A is invertible and $B = A^{-1}$;
- (b) if $U \neq I_n$, then A is not invertible.

Proof. By Theorem 1.11.9, we have that A is invertible iff $\mathrm{RREF}(A) = I_n$, and since $\left[\begin{array}{c:c} U & B \end{array}\right] = \mathrm{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right])$, we have that $\mathrm{RREF}(A) = U$.

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- (a) if $U = I_n$, then A is invertible and $B = A^{-1}$;
- (b) if $U \neq I_n$, then A is not invertible.

Proof. By Theorem 1.11.9, we have that A is invertible iff $\mathrm{RREF}(A) = I_n$, and since $\left[\begin{array}{c:c} U & B \end{array}\right] = \mathrm{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right])$, we have that $\mathrm{RREF}(A) = U$.

So, if $U \neq I_n$, then A is not invertible; this proves (b) holds.

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- ⓐ if $U = I_n$, then A is invertible and $B = A^{-1}$;
- ⓑ if $U \neq I_n$, then A is not invertible.

Proof. By Theorem 1.11.9, we have that A is invertible iff $\text{RREF}(A) = I_n$, and since $\left[\begin{array}{c:c} U & B \end{array}\right] = \text{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right])$, we have that $\text{RREF}(A) = U$.

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Assume now that $U = I_n$, so that A is invertible.

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So, if $U \neq I_n$, then A is not invertible; this proves (b) holds.

Assume now that $U = I_n$, so that A is invertible. To prove (a), it now remains to show that $B = A^{-1}$.

Proof (continued). Reminder: $U = I_n$, A is invertible, RREF$\left(\left[\begin{array}{c:c} A & I_n \end{array} \right] \right) = \left[\begin{array}{c:c} I_n & B \end{array} \right]$. WTS $B = A^{-1}$.

Proof (continued). Reminder: $U = I_n$, A is invertible, $\mathrm{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right]) = \left[\begin{array}{c:c} I_n & B \end{array}\right]$. WTS $B = A^{-1}$.

Since $\left[\begin{array}{c:c} A & I_n \end{array}\right] \sim \left[\begin{array}{c:c} I_n & B \end{array}\right]$, Theorem 1.11.13 guarantees that there exists an invertible matrix $C \in \mathbb{F}^{n\times n}$ s.t.
$C\left[\begin{array}{c:c} A & I_n \end{array}\right] = \left[\begin{array}{c:c} I_n & B \end{array}\right]$.

Proof (continued). Reminder: $U = I_n$, A is invertible, $\text{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right]) = \left[\begin{array}{c:c} I_n & B \end{array}\right]$. WTS $B = A^{-1}$.

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$C\left[\begin{array}{c:c} A & I_n \end{array}\right] = \left[\begin{array}{c:c} I_n & B \end{array}\right]$. But note that
$C\left[\begin{array}{c:c} A & I_n \end{array}\right] = \left[\begin{array}{c:c} CA & C \end{array}\right]$.

Proof (continued). Reminder: $U = I_n$, A is invertible, $\text{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right]) = \left[\begin{array}{c:c} I_n & B \end{array}\right]$. WTS $B = A^{-1}$.

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Proof (continued). Reminder: $U = I_n$, A is invertible, $\text{RREF}(\left[\begin{array}{c|c} A & I_n \end{array}\right]) = \left[\begin{array}{c|c} I_n & B \end{array}\right]$. WTS $B = A^{-1}$.

Since $\left[\begin{array}{c|c} A & I_n \end{array}\right] \sim \left[\begin{array}{c|c} I_n & B \end{array}\right]$, Theorem 1.11.13 guarantees that there exists an invertible matrix $C \in \mathbb{F}^{n \times n}$ s.t.
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$C\left[\begin{array}{c|c} A & I_n \end{array}\right] = \left[\begin{array}{c|c} CA & C \end{array}\right]$. So,
$\left[\begin{array}{c|c} CA & C \end{array}\right] = C\left[\begin{array}{c|c} A & I_n \end{array}\right] = \left[\begin{array}{c|c} I_n & B \end{array}\right]$, which in turn implies that $CA = I_n$ and $C = B$, and consequently, $BA = I_n$.

Proof (continued). Reminder: $U = I_n$, A is invertible, $\text{RREF}(\left[\begin{array}{c|c} A & I_n \end{array}\right]) = \left[\begin{array}{c|c} I_n & B \end{array}\right]$. WTS $B = A^{-1}$.

Since $\left[\begin{array}{c|c} A & I_n \end{array}\right] \sim \left[\begin{array}{c|c} I_n & B \end{array}\right]$, Theorem 1.11.13 guarantees that there exists an invertible matrix $C \in \mathbb{F}^{n\times n}$ s.t. $C\left[\begin{array}{c|c} A & I_n \end{array}\right] = \left[\begin{array}{c|c} I_n & B \end{array}\right]$. But note that $C\left[\begin{array}{c|c} A & I_n \end{array}\right] = \left[\begin{array}{c|c} CA & C \end{array}\right]$. So, $\left[\begin{array}{c|c} CA & C \end{array}\right] = C\left[\begin{array}{c|c} A & I_n \end{array}\right] = \left[\begin{array}{c|c} I_n & B \end{array}\right]$, which in turn implies that $CA = I_n$ and $C = B$, and consequently, $BA = I_n$. But we already saw that A is invertible, and so Proposition 1.11.3 (below) guarantees that $A^{-1} = B$. □

Proposition 1.11.3

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Assume that A is invertible and that $AB = I_n$ or $BA = I_n$. Then $A^{-1} = B$.

Theorem 1.11.4

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$ be a square matrix, and set $\left[\begin{array}{c:c} U & B \end{array}\right] = \mathrm{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right])$, where each of U and B has n columns. Then

ⓐ if $U = I_n$, then A is invertible and $B = A^{-1}$;

ⓑ if $U \neq I_n$, then A is not invertible.

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  - Essentially, it is a long list of references to results that we have already proven.
- Later in the course, we will extend the Invertible Matrix Theorem (i.e. add more equivalent statements) to it.

- The Invertible Matrix Theorem (next two slides) gives a long list of statements that are equivalent to a square matrix being invertible.
    - The theorem is so long that it does not fit onto one slide!
- This theorem is essentially a summary of the results that we have proven so far.
    - The proof of the Invertible Matrix Theorem is given in the Lecture Notes.
    - Essentially, it is a long list of references to results that we have already proven.
- Later in the course, we will extend the Invertible Matrix Theorem (i.e. add more equivalent statements) to it.
- Importantly, the Invertible Matrix Theorem applies only to **square** matrices, and may not be applies to non-square matrices.

The Invertible Matrix Theorem (version 1)

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be a **square** matrix. Further, let $f : \mathbb{F}^n \to \mathbb{F}^n$ be given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$.[a] Then the following are equivalent:

- (a) A is invertible (i.e. A has an inverse);
- (b) A^T is invertible;
- (c) $\text{RREF}(A) = I_n$;
- (d) $\text{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right]) = \left[\begin{array}{c:c} I_n & B \end{array}\right]$ for some matrix $B \in \mathbb{F}^{n \times n}$;
- (e) $\text{rank}(A) = n$;
- (f) $\text{rank}(A^T) = n$;
- (g) A is a product of elementary matrices;

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[a] Since f is a matrix transformation, Proposition 1.10.4 guarantees that f is linear. Moreover, A is the standard matrix of f.

The Invertible Matrix Theorem (version 1) - continued

(h) the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (i.e. the solution $\mathbf{x} = \mathbf{0}$);

(i) there exists some vector $\mathbf{b} \in \mathbb{F}^n$ s.t. the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;

(j) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;

(k) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has at most one solution;

(l) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent;

(m) f is one-to-one;

(n) f is onto;

(o) f is an isomorphism.