Linear Algebra 1: HW#10

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due Friday, January 17, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (including any row reduction that you may need to perform in order to solve the exercise/problem). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Problem 1 (25 points). *Consider the following set of vectors (matrices) in* $\mathbb{R}^{3\times 3}$*:*

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & \pi^5 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & \pi^5 \\ \pi^6 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & \pi^5 \\ \pi^6 & \pi^7 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & \pi^5 \\ \pi^6 & \pi^7 & \pi^8 \end{bmatrix} \right\}.$$

(a) Determine whether $\mathcal{C}$ *is a linearly independent set in* $\mathbb{R}^{3\times 3}$.

(b) Determine whether $\mathcal{C}$ *is a spanning set of* $\mathbb{R}^{3\times 3}$.

(c) Determine whether $\mathcal{C}$ *is a basis of* $\mathbb{R}^{3\times 3}$.

> ***Remark/Hint:*** *You shouldn't need to do any row reducing in this problem.*

Problem 2 (25 points). *Consider the following polynomials in* $\mathbb{P}^4_{\mathbb{Z}_2}$*:*

- $p_1(x) = x^4 + x^3 + 1$;
- $p_2(x) = x^3 + x^2$;
- $p_3(x) = x^4 + x^2 + 1$;
- $p_4(x) = x^3 + x^2 + x + 1$;
- $p_5(x) = x^4 + x^3 + x$.

Find a basis $\mathcal{B}_U$ *of* $U := Span\big(p_1(x), \ldots, p_5(x)\big)$*, and extend it to a basis* $\mathcal{B}$ *of* $\mathbb{P}^4_{\mathbb{Z}_2}$*. Then, for each* $i \in \{1, \ldots, 5\}$*, compute the coordinate vector* $\big[\ p_i(x)\ \big]_{\mathcal{B}}$.

Problem 3 (25 points). *Let* U *and* V *be vector spaces over a field* $\mathbb{F}$ *such that* U *is finite-dimensional and* $dim(U) \leq dim(V)$*. Prove that there exists a one-to-one linear function* $f : U \to V$.

> ***Remark:*** *This is essentially the converse of Theorem 4.2.14 of the Lecture Notes.*

Problem 4 (25 points). *Give a detailed proof of Corollary 4.3.3 of the Lecture Notes.*

> ***Remark:*** *A proof outline is given in the Lecture Notes. Your job is to turn this outline into a detailed proof.*