Linear Algebra 1: HW#9

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due Monday, January 6, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (including any row reduction that you may need to perform in order to solve the exercise/problem). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (30 points)**.** *Consider the following vectors, with entries understood to be in* $\mathbb{Z}_3$*:*

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

Find a basis $\mathcal{A}$ *of* $Span(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$*, and find a basis* $\mathcal{C}$ *of* $\mathbb{Z}_3^4$ *that extends* $\mathcal{A}$ *(i.e. that satisfies* $\mathcal{A} \subseteq \mathcal{C}$*). Then, for each* $i \in \{1, 2, 3, 4\}$*, compute the coordinate vector* $\left[\ \mathbf{a}_i\ \right]_{\mathcal{C}}$*.*

> ***Remark:*** *See subsection 3.3.5 of the Lecture Notes.*

Problem 1 (60 points)**.** *Consider the linear function* $f : \mathbb{Z}_3^4 \to \mathbb{Z}_3^6$ *whose standard matrix is*

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix},$$

and consider the following vectors with entries in $\mathbb{Z}_3$*:*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

Set $U := Span(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ *and* $V := Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$*.*[1]

(a) [25 points] Compute a basis of $f[U]$*.*

(b) [35 points] Compute a basis of $f^{-1}[V]$*.*

> ***Remark:*** *See subsection 4.2.4 of the Lecture Notes.*

[1] So, U is a subspace of $\mathbb{Z}_3^4$, and V is a subspace of $\mathbb{Z}_3^6$.

Problem 2 (10 points)**.** *Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and set $n := dim(V)$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in V$.*[2] *Prove that the following are equivalent:*

(1) $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ *is a linearly independent set;*

(2) $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ *is a spanning set of V;*

(3) $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ *is a basis of V.*

> ***Hint:*** *This is a nearly immediate consequence of a certain result (which one?) from section 3.2 of the Lecture Notes. Your job is to figure out what result that is, and then to write a short paragraph explaining why that result indeed implies that (1), (2) and (3) are equivalent.*

[2] So, we have n vectors in an n-dimensional vector space.