Linear Algebra 1: HW#8

Irena Penev
Winter 2024

due Monday, December 23, 2024, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (including any row reduction that you may need to perform in order to solve the exercise/problem). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (20 points). *Consider the matrix*

$$A \;:=\; \begin{bmatrix} 2 & 0 & 1 & 2 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 & 2 & 2 & 1 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. Compute each of the following:

(a) a basis of Col(A);

(b) a basis of Row(A);

(c) a basis of Nul(A).

Problem 1 (20 points). *Prove Proposition 3.2.12 of the Lecture Notes (see page 230).*

Problem 2 (20 points). *Complete the proof of Theorem 3.2.23 of the Lecture Notes by showing that the set*

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{u}_1, \ldots, \mathbf{u}_{m-p}, \mathbf{w}_1, \ldots, \mathbf{w}_{n-p}\}$$

is indeed a basis of $U + W$ (see page 240 of the Lecture Notes).

Problem 3 (20 points). *Prove Theorem 3.2.24 of the Lecture Notes (see page 241).*

Problem 4 (20 points). *Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U be a subspace of V. Prove that there exists a subspace W of V such that $V = U \oplus W$.*