Linear Algebra 1: HW#6

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due Friday, November 29, 2024, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (including any row reduction that you may need to perform in order to solve the exercise/problem). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (20 points). *Consider the following permutation in* S_8*:*

$$\pi \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 6 & 7 & 5 & 4 & 8 \end{pmatrix}.$$

(a) [4 points] Compute the disjoint cycle decomposition of π*. Include one-element cycles (if there are any).*

(b) [3 points] Express π *as a composition of transpositions.*

(c) [3 points] List all inversions of π*. How many inversions does* π *have?*

(d) [6 points] Compute sgn(π) *in three different ways:*

(d.1) using the disjoint cycle decomposition of π*, i.e. using your answer from part (a);*

(d.2) using transpositions, i.e. using your answer from part (b);

(d.2) using inversions, i.e. using your answer from part (c).

(e) [4 points] Compute π^{-1}*. Give both the table form of* π^{-1} *and the disjoint cycle decompositions of* π^{-1}*.*

Exercise 2 (10 points). *Let* $\sigma = (143)(658)$ *and* $\pi = (254)(3768)$ *be permutations in* S_8*. Compute the disjoint cycle decomposition of* $\sigma \circ \pi$ *and* $\pi \circ \sigma$ *(**omit** cycles of length one, if any).*

Problem 1 (40 points). *Let* $(G, \cdot)$ *be a finite group, with identity element* 1*. For any element* $g \in G$*, we denote the inverse of* g *by* g^{-1}*, we set* $g^0 = 1$*, and for all non-negative integers* m*, we set* $g^{m+1} = g^m g$*. So, for any* $g \in G$ *and any positive integer* m*, we have that* $g^m = \underbrace{g \cdot \dots \cdot g}_{m}$*. Now, let* $a \in G$*.*

(a) [15 points] Prove that there exists a positive integer n *such that* $a^n = 1$*.*

> ***Hint:*** *Consider the sequence* $a^0, a^1, a^2, a^3, a^4, \dots$*, and explain why some two terms of this sequence are equal. Now what?*

From now on, let n *be the **smallest** positive integer such that* $a^n = 1$*. (Part (a) guarantees that such an* n *exists.) Set* $H := \{a^1, a^2, \dots, a^n\}$*.*

(b) [15 points] Prove that H *is a subgroup of* G*.*

(c) [10 points] Prove that $|H| = n$*, that is, prove that* $a^1, a^2, \dots, a^n$ *are pairwise distinct.*

> ***Remark:*** *Together with Exercise 1 from Tutorial 7, parts (b) and (c) guarantee that* $n \mid |G|$ *(i.e.* n *divides* $|G|$*).*

Problem 2 (30 points). *Let $n \geq 2$ be an integer. Prove that for any permutation $\pi \in S_n$, we have that*

$$sgn(\pi) \quad = \quad (-1)^{k_{even}},$$

where k_{even} is the number of even cycles in the disjoint cycle decomposition of π.[1]

[1] An *even cycle* is a cycle that contains an even number of elements, and an *odd cycle* is a cycle that contain an odd number of elements. For example, the permutation (1)(27)(39)(4568) in S_9 has three even cycles (and one odd cycle).