Linear Algebra 1: HW#5

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due Friday, November 22, 2024, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (including any row reduction that you may need to perform in order to solve the exercise/problem). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (20 points). *For each of the following two matrices, determine whether the matrix is invertible, and if so, compute its inverse:*

(a) $A = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$, *with entries in* $\mathbb{Z}_3$*;*

(b) $B = \begin{bmatrix} 2 & -5 & 2 \\ 3 & 1 & -2 \\ -1 & 11 & 2 \end{bmatrix}$, *with entries in* $\mathbb{R}$.

Problem 1 (20 points). *Consider the matrix*

$$A \;:=\; \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. *Either express* A *as a product of elementary matrices, or explain why this is not possible. (If it is possible, then try not to use too many elementary matrices.)*

Problem 2 (20 points). *Consider the matrices*

$$A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. *Either compute an invertible matrix* $C \in \mathbb{Z}_3^{3\times 3}$ *such that* $B = CA$, *or prove that no such matrix* C *exists.*

Problem 3 (20 points). *Consider the following vectors, with entries understood to be in* $\mathbb{Z}_2$*:*

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Determine if there exists a linear function $f : \mathbb{Z}_2^3 \to \mathbb{Z}_2^4$ *that satisfies the property that*

$$f(\mathbf{b}_i) = \mathbf{c}_i \quad \text{for all } i \in \{1, 2, 3, 4\}.$$

If such a linear function f *exists, determine whether it is unique, and if it is unique, find a formula for* f.

Remark: *If the linear function f exists and is unique, then your formula for it should be of the form*

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \qquad \textit{for all } x_1, x_2, x_3 \in \mathbb{Z}_2,$$

where the question marks are replaced with the appropriate values. If the linear function f does not exist, or if it exists but is not unique, then you do not need to find such a formula.

Hint: *This problem essentially boils down to solving a matrix equation of the form $XA = B$. Several examples very similar to this problem are given in subsection 1.10.4 of the Lecture Notes.*

Problem 4 (20 points)**.** *Prove parts (b) and (d) of Proposition 2.2.4 from the Lecture Notes.*

Hint: *Imitate the proofs of parts (a) and (c).*