Linear Algebra 1: HW#4

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due Friday, November 8, 2024, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (including any row reduction that you may need to perform in order to solve the exercise/problem). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a]If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b]For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (10 points). *Consider the linear function* $f : \mathbb{R}^2 \to \mathbb{R}^4$ *given by*

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + 3x_2 \\ 2x_1 - 7x_2 \\ 3x_1 \\ x_1 - x_2 \end{bmatrix}$$

for all $x_1, x_2 \in \mathbb{R}$. *Compute the standard matrix of* f. *(You may assume that* f *is indeed linear, i.e. you do not need to prove this.)*

Exercise 2 (10 points). *Consider the matrices*

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. *Let* $f : \mathbb{Z}_3^4 \to \mathbb{Z}_3^2$ *be given by* $f(\mathbf{x}) = A\mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{Z}_3^4$, *and let* $g : \mathbb{Z}_3^2 \to \mathbb{Z}_3^4$ *be given by* $g(\mathbf{x}) = B\mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{Z}_3^2$.[1] *Compute the standard matrices of the linear functions* $f \circ g$ *and* $g \circ f$ *(and make sure you indicate which is which).*[2]

Problem 1 (30 points). *Let* $f : \mathbb{R}^m \to \mathbb{R}^n$ *be a function that satisfies the following property:*

> *for all vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ *and scalars* $\alpha, \beta \in \mathbb{R}$, *we have that* $f(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v})$.

Prove that the function f *is linear.*

> ***Remark:*** *In fact, this is true if* $\mathbb{R}$ *is replaced by* ***any*** *field* $\mathbb{F}$ *(and the proof remains pretty much identical). However, for the sake of simplicity, you are only asked to prove it for* $\mathbb{R}$.

Problem 2 (25 points). *Consider the matrices*

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

with entries understood to be in $\mathbb{Z}_2$. *Solve the matrix equation* $AX = B$. *How many solutions does the equation* $AX = B$ *have?*

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[1] Note that this means that f and g are matrix transformations and are therefore linear (by Proposition 1.10.4 of the Lecture Notes). Moreover, A and B are the standard matrices of f and g, respectively (by the definition of a standard matrix).

[2] The fact that $f \circ g$ and $g \circ f$ are indeed linear follows from Proposition 1.10.13(c) of the Lecture Notes.

Problem 3 (25 points). *Consider the matrices*

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. *Solve the matrix equation* $XA = B$. *How many solutions does the equation* $XA = B$ *have?*