Linear Algebra 1: HW#3

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due Friday, November 1, 2024, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (including any row reduction that you may need to perform in order to solve the exercise/problem). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (10 points)**.** *Consider the following matrices, with entries understood to be in* $\mathbb{Z}_2$*:*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Compute the matrices AB *and* BA *(and make sure you indicate which is which).*[1] *(Just write the final answer. You do not need to "show your work" in this problem.)*

Exercise 2 (15 points)**.** *Consider the following matrix and vector, with entries understood to be in* $\mathbb{Z}_3$*:*

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$*, and state how many solutions it has.*

Exercise 3 (15 points)**.** *Consider the following matrix and vector, with entries understood to be in* $\mathbb{Z}_2$*:*

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$*, and state how many solutions it has.*

Problem 1 (30 points)**.** *Let* a *and* b *be real constants. For what values of the real constant* c *is the vector* $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ *a linear combination of the vectors* $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$*? Make sure you prove that your answer is correct.*

[1] Since you are working in $\mathbb{Z}_2$, all entries in your matrices AB and BA should be 0 or 1.

Problem 2 (30 points). *Consider the following vectors in* $\mathbb{Z}_3^4$*:*

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

For each of the vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$*, determine if it can be expressed as a linear combination of the vectors* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ *(that is, if it belongs to Span(*$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$*)), and if so, express it as such a linear combination, and explain whether your answer is unique.*