Linear Algebra 1:
HW#1

Irena Penev
Winter 2024

due Friday, October 11, 2024, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Exercise 1 (10 points)**.** *Compute the following:*

(a) $1-1+1-1-1$ *in* $\mathbb{Z}_2$*;*

(b) 2^5 *in* $\mathbb{Z}_3$*;*

(c) $(-1)^5$ *in* $\mathbb{Z}_3$*;*

(d) $2 \cdot 2 + 3 \cdot 3 - 2 \cdot 3$ *in* $\mathbb{Z}_5$*;*

(e) $1^2 + 2^2 + 3^2 + 4^2$ *in* $\mathbb{Z}_5$*.*

In this exercise, you should just write the final answer (there is no need to show your work). If you are asked to compute in $\mathbb{Z}_n$*, then your answer should be one of the following:* $0, 1, \ldots, n-1$*. (In particular, your final answer should contain* ***no minuses****.)*

Problem 1 (15 points)**.** *Compute* 57^{102} *in* $\mathbb{Z}_{101}$*. Make sure you justify your answer. Your final answer should be one of the following:* $0, 1, 2, \ldots, 100$*.*

Hint: *Start by using Fermat's Little Theorem.*

Remark: All the problems below should be solved using **induction**. **Solutions that do not use induction will receive no credit.**

Problem 2 (25 points)**.** *Compute the last* ***two digits*** *of the number* 12345^{12345} *(and prove that your answer is correct).*

Hint: *This is similar to Example 0.2.5 from the Lecture Notes.*

Problem 3 (25 points)**.** *The* Fibonacci numbers *are defined as follows:*

- $F(1) = F(2) = 1$*;*
- $F(n+2) = F(n) + F(n+1)$ *for all positive intgers* n*.*

Prove that for all positive integers n*, there are exactly* $F(n+1)$ *many ways to cut up a rectangle of size* $2 \times n$ *into rectangles of size* 1×2*.*

Problem 4 (25 points)**.** *Show that any collection of at least 5 cities can be connected via one-way flights*[1] *in such a way that any city is reachable from any other city with at most one layover.*

[1] So, for any two cities, A and B, at most one of the following (direct) flights is possible:

- from A to B;
- from B to A.