Linear Algebra 1: Tutorial 9

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Exercise 1. *Let $\mathbb{F}$ be a field. Find the dimension of each of the following vector spaces (over $\mathbb{F}$):*

(a) $\mathbb{P}^n_{\mathbb{F}}$, *where n is a non-negative integer;*

(b) $\{p(x) \in \mathbb{P}^n_{\mathbb{F}} \mid p(0) = 0\}$, *where n is a non-negative integer;*

(c) $\mathbb{F}^{n \times m}$, *where n and m are positive integers.*

> ***Hint:*** *Experiment with small values of n and m first, and then generalize.*

Exercise 2. *For each of the following vector spaces V and (ordered) sets $\mathcal{B}$, explain why $\mathcal{B}$ is* ***not*** *a basis of V. Try to compute as little as possible.*

(a) $V = \mathbb{R}^{2\times 2}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}, \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \right\}$;

(b) $V = \mathbb{R}^{2\times 2}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$;

(c) $V = \mathbb{R}^{2\times 2}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$;

(d) $V = \mathbb{P}^3_{\mathbb{Z}_2}$, $\mathcal{B} = \left\{ x+1, x^2+x, 0, x^3 \right\}$;

(e) $V = \mathbb{P}^2_{\mathbb{Z}_3}$, $\mathcal{B} = \left\{ 2x+1, x+2, 1 \right\}$.

Exercise 3. *Find bases of the column, row, and null space of each of the matrices below. Moreover, compute the rank and the nullity of those matrices.*

(a) $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, *with entries understood to be in $\mathbb{Z}_2$;*

(b) $B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix}$, *with entries understood to be in* $\mathbb{Z}_3$*;*

(c) $C = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -3 \\ 0 & 7 & 0 \end{bmatrix}$, *with entries understood to be in* $\mathbb{R}$*.*