Linear Algebra 1: Tutorial 8

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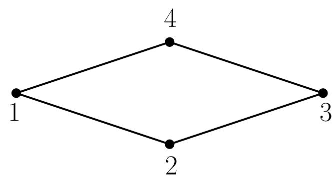
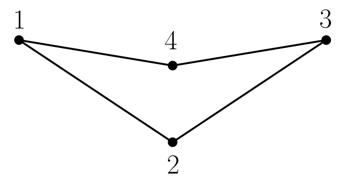
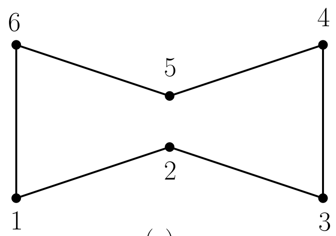
Exercise 3 from Tutorial 7. *Compute the group of symmetries for each of the polygons below.*



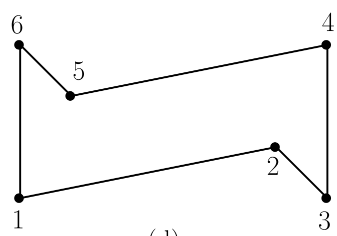
(a)



(b)



(c)



(d)

Exercise 4 from Tutorial 7. *Let n and k be positive integers such that $k \leq n$. What is the probability that in a random permutation in S_n, the number 1 is in a cycle of length k?*

Definition. *A* finite basis *(or simply* basis*) of a vector space V over a field $\mathbb{F}$ is a set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of vectors in V that satisfies the following two conditions:*

1. *$\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent in V;*
2. *$\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a spanning set of V, i.e. Span$(\mathbf{v}_1, \dots, \mathbf{v}_k) = V$.*

Proposition 3.2.6 from the Lecture Notes. *Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ ($m \geq 1$) be vectors in $\mathbb{F}^n$. Set $A := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$. Then $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ is a basis of $\mathbb{F}^n$ if and only if rank$(A) = n = m$ (i.e. A is a square matrix of full rank). In particular, every basis of $\mathbb{F}^n$ contains exactly n vectors.*

Proof. By Proposition 3.2.1, vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ are linearly independent if and only if $\text{rank}(A) = m$, and by Proposition 3.1.10, vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ span $\mathbb{F}^n$ if and only if $\text{rank}(A) = n$. So, $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ is a basis of $\mathbb{F}^n$ if and only if $\text{rank}(A) = m = n$. □

Exercise 1. *For each of the given vector spaces V and sets of vectors $\mathcal{B}$, determine whether $\mathcal{B}$ is (1) a linearly independent set in V, (2) a spanning set of V, and (3) a basis of V.*

(a) $V = \mathbb{Z}_2^5$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$;

(b) $V = \mathbb{Z}_3^2$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$;

(c) $V = \mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ \pi \\ \pi^2 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 29 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$;

(d) $V = \mathbb{R}^{2\times 2}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

(e) $V = \mathbb{P}_{\mathbb{R}}^3$, $\mathcal{B} = \left\{ 1, 1+x, 1+x+x^2, 1+x+x^2+x^3 \right\}$.