Linear Algebra 1: Tutorial 3

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Exercise 1. *Solve the linear system below, where k is some fixed constant. (The coefficients are assumed to be in $\mathbb{R}$.)*

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 3x_1 & + & 4x_2 & + & 2x_3 & = & 3 \\ 5x_1 & + & 8x_2 & + & (k-3)^2 x_3 & = & k \end{array}$$

> ***Remark:*** *Your solutions will depend on k. In fact, the* ***number*** *of solutions will also depend on k. You will need to figure out for which (if any) k the system has no solutions, for which it has a unique solution, and for which it has infinitely many solutions.*

Exercise 2. *Consider the matrix and vector below, with entries understood to be in $\mathbb{Z}_3$.*

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$. How many solutions does this equation have?

Exercise 3. *Consider the matrix and vector below, with entries understood to be in $\mathbb{R}$.*

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 3 & 2 \\ -2 & 6 & -2 \\ 3 & -9 & 0 \\ 1 & -3 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$. How many solutions does this equation have?

Exercise 4. *Consider the matrix and vector below, with entries understood to be in* $\mathbb{Z}_2$.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$. *How many solutions does this equation have?*

Exercise 5. *Consider the following vectors in* $\mathbb{Z}_2^4$:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

For each of the vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}$, *determine if it can be expressed as a linear combination of the vectors* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ *(that is, if it belongs to Span*$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$*), and if so, express it as such a linear combination, and explain whether your answer is unique.*