Linear Algebra 1

Lecture #12

Affine subspaces and affine functions

Irena Penev

January 8, 2024

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  - To avoid any confusion, in this chapter, we will not use the term "subspace" and will instead always write either "linear subspace" or "affine subspace."
  - However, in subsequent lectures (next semester), we will again use the term "subspace" to mean "linear subspace."

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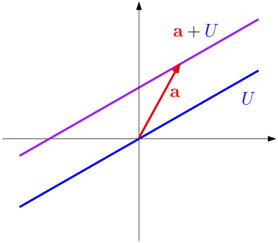
Definition

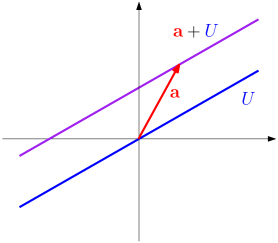
An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form

$$\mathbf{a} + U \quad := \quad \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\},$$

where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.



- Thus, an affine subspace of V is obtained by shifting a linear subspace U of V by some vector $\mathbf{a}$.



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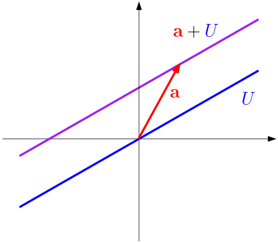

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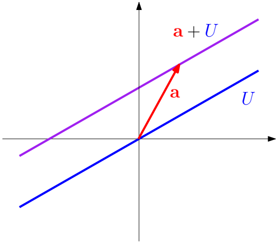

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  - Every linear subspace U of V is also an affine subspace of V, since $U = \mathbf{0} + U$.
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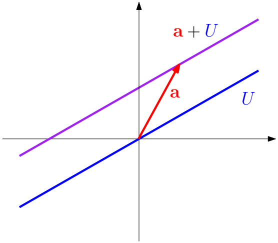

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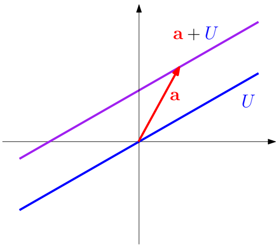

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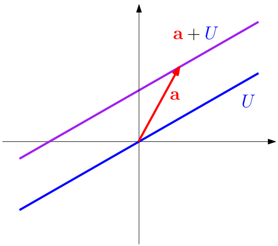
    - Moreover, as we shall see, linear subspaces of V are precisely those affine subspaces of V that contain $\mathbf{0}$ (see Corollary 5.1.2).
  - V is an affine subspace of itself (because V is a linear subspace of itself).
  - For every vector $\mathbf{a} \in V$, $\{\mathbf{a}\}$ is an affine subspace of V, since $\{\mathbf{a}\} = \mathbf{a} + \{\mathbf{0}\}$ and $\{\mathbf{0}\}$ is a linear subspace of V.



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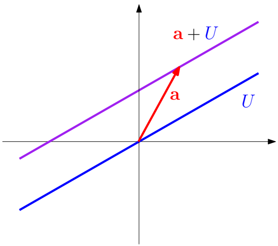

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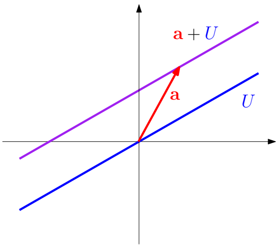
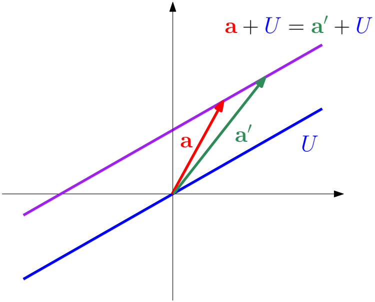
  - As we know, linear subspaces of $\mathbb{R}^n$ are $\{\mathbf{0}\}$, lines through the origin, planes through the origin, and higher dimensional generalizations.



- **Geometric considerations:**
  - As we know, linear subspaces of $\mathbb{R}^n$ are $\{\mathbf{0}\}$, lines through the origin, planes through the origin, and higher dimensional generalizations.
  - So, affine subspaces of $\mathbb{R}^n$ are $\{\mathbf{a}\}$ (for any vector $\mathbf{a} \in \mathbb{R}^n$), lines, planes, and higher dimensional generalizations (these lines, planes, and higher dimensional generalizations may, but need not, pass through the origin).

- As Theorem 5.1.1 (next slide) shows, for an affine subspace $M = \mathbf{a} + U$ of a vector space V over a field $\mathbb{F}$ (where $\mathbf{a}$ is a vector and U a linear subspace of V), the vector $\mathbf{a}$ need not be unique (indeed, it can be any vector in M), but the linear subspace U is unique (it depends only on M, and not on the vector $\mathbf{a}$).



Theorem 5.1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

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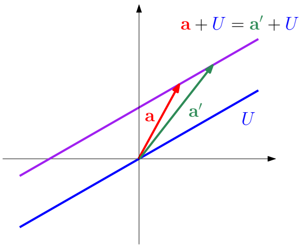

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Proof of (a).

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Proof of (a). Since U is a linear subspace of V, Theorem 3.1.7 guarantees that $\mathbf{0} \in U$, and consequently, $\mathbf{a} = \mathbf{a} + \mathbf{0} \in \mathbf{a} + U = M$.

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Proof of (b).

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Proof of (b). Fix $\mathbf{a}' \in M$.

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Proof of (b) (continued). Reminder: $\mathbf{a}' \in M$, $\mathbf{u}' \in U$, $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

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Let us now show that $\mathbf{a}' + U \subseteq M$. Fix $\mathbf{u} \in U$; we must show that $\mathbf{a}' + \mathbf{u} \in M$. But note that $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u}$.

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b. for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;
c. for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$;
d. for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$.

Proof of (b) (continued). Reminder: $\mathbf{a}' \in M$, $\mathbf{u}' \in U$, $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us now show that $\mathbf{a}' + U \subseteq M$. Fix $\mathbf{u} \in U$; we must show that $\mathbf{a}' + \mathbf{u} \in M$. But note that $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u}$. Since $\mathbf{u}', \mathbf{u} \in U$, and U is a linear subspace of V, we have that $\mathbf{u}' + \mathbf{u} \in U$;

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Proof of (b) (continued). Reminder: $\mathbf{a}' \in M$, $\mathbf{u}' \in U$, $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us now show that $\mathbf{a}' + U \subseteq M$. Fix $\mathbf{u} \in U$; we must show that $\mathbf{a}' + \mathbf{u} \in M$. But note that $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u}$. Since $\mathbf{u}', \mathbf{u} \in U$, and U is a linear subspace of V, we have that $\mathbf{u}' + \mathbf{u} \in U$; consequently, $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u} \in \mathbf{a} + U = M$.

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Proof of (b) (continued). Reminder: $\mathbf{a}' \in M$, $\mathbf{u}' \in U$, $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us now show that $\mathbf{a}' + U \subseteq M$. Fix $\mathbf{u} \in U$; we must show that $\mathbf{a}' + \mathbf{u} \in M$. But note that $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u}$. Since $\mathbf{u}', \mathbf{u} \in U$, and U is a linear subspace of V, we have that $\mathbf{u}' + \mathbf{u} \in U$; consequently, $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u} \in \mathbf{a} + U = M$. This proves that $\mathbf{a}' + U \subseteq M$.

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Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

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(c) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$;

(d) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$.

Proof of (b) (continued). Reminder: $\mathbf{a}' \in M$, $\mathbf{u}' \in U$, $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

We have now proven the following:

- $M \subseteq \mathbf{a}' + U$;
- $\mathbf{a}' + U \subseteq M$.

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- (a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);
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Proof of (b) (continued). Reminder: $\mathbf{a}' \in M$, $\mathbf{u}' \in U$, $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$.
WTS $M = \mathbf{a}' + U$.

We have now proven the following:

- $M \subseteq \mathbf{a}' + U$;
- $\mathbf{a}' + U \subseteq M$.

This proves that $M = \mathbf{a} + U$, which is what we needed.

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Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

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Proof of (c).

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Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

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Proof of (c). Fix a vector $\mathbf{a}'$ and a linear subspace U' of V s.t. $M = \mathbf{a}' + U'$. WTS $U' = U$.

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Proof of (c). Fix a vector $\mathbf{a}'$ and a linear subspace U' of V s.t. $M = \mathbf{a}' + U'$. WTS $U' = U$.

By (a), we have that $\mathbf{a}' \in M$,

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Proof of (c). Fix a vector $\mathbf{a}'$ and a linear subspace U' of V s.t. $M = \mathbf{a}' + U'$. WTS $U' = U$.

By (a), we have that $\mathbf{a}' \in M$, and so by (b), we have that $M = \mathbf{a}' + U$.

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Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

- (a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);
- (b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;
- (c) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$;
- (d) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$.

Proof of (c). Fix a vector $\mathbf{a}'$ and a linear subspace U' of V s.t. $M = \mathbf{a}' + U'$. WTS $U' = U$.

By (a), we have that $\mathbf{a}' \in M$, and so by (b), we have that $M = \mathbf{a}' + U$. So, $\mathbf{a}' + U' = \mathbf{a}' + U$, and we deduce that $U' = U$.

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Proof of (d).

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Proof of (d). Fix $\mathbf{b} \in V \setminus M$. WTS $M \cap (\mathbf{b} + U) = \emptyset$. Suppose otherwise, and fix $\mathbf{x} \in M \cap (\mathbf{b} + U)$.

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(d) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$.

Proof of (d). Fix $\mathbf{b} \in V \setminus M$. WTS $M \cap (\mathbf{b} + U) = \emptyset$. Suppose otherwise, and fix $\mathbf{x} \in M \cap (\mathbf{b} + U)$. Since $\mathbf{x} \in M = \mathbf{a} + U$, there exists some $\mathbf{u}_1 \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}_1$; on the other hand, since $\mathbf{x} \in \mathbf{b} + U$, there exists some $\mathbf{u}_2 \in U$ s.t. $\mathbf{x} = \mathbf{b} + \mathbf{u}_2$. So, $\mathbf{a} + \mathbf{u}_1 = \mathbf{b} + \mathbf{u}_2$, and it follows that $\mathbf{b} = \mathbf{a} + (\mathbf{u}_1 - \mathbf{u}_2)$.

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Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

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Proof of (d) (continued). Reminder: $\mathbf{b} = \mathbf{a} + (\mathbf{u}_1 - \mathbf{u}_2)$.

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Proof of (d) (continued). Reminder: $\mathbf{b} = \mathbf{a} + (\mathbf{u}_1 - \mathbf{u}_2)$.

Since $\mathbf{u}_1, \mathbf{u}_2 \in U$, and since U is a linear subspace of V, we have that $\mathbf{u}_1 - \mathbf{u}_2 \in U$;

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(c) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$;

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Proof of (d) (continued). Reminder: $\mathbf{b} = \mathbf{a} + (\mathbf{u}_1 - \mathbf{u}_2)$.

Since $\mathbf{u}_1, \mathbf{u}_2 \in U$, and since U is a linear subspace of V, we have that $\mathbf{u}_1 - \mathbf{u}_2 \in U$; consequently, $\mathbf{b} = \mathbf{a} + (\mathbf{u}_1 - \mathbf{u}_2) \in \mathbf{a} + U = M$, contrary to the fact that $\mathbf{b} \in V \setminus M$. $\square$

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Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);

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(c) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$;

(d) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$.

- Given a vector space V over a field $\mathbb{F}$, we define the *dimension* of an affine subspace $M = \mathbf{a} + U$ of V (where $\mathbf{a}$ is a vector and U a linear subspace of V) to be

$$\dim(M) \quad := \quad \dim(U).$$

By Theorem 5.1.1(c), this is well defined.

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- (d) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$.

Corollary 5.1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

- (i) U is a linear subspace of V;
- (ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Corollary 5.1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

(i) U is a linear subspace of V;

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Proof.

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1. (i) U is a linear subspace of V;
2. (ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds.

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Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

1. (i) U is a linear subspace of V;
2. (ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then $\mathbf{0} \in U$ (by Theorem 3.1.7), and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

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Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then $\mathbf{0} \in U$ (by Theorem 3.1.7), and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

Suppose now that (ii) holds.

Corollary 5.1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

1. (i) U is a linear subspace of V;
2. (ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then $\mathbf{0} \in U$ (by Theorem 3.1.7), and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

Suppose now that (ii) holds. Since U is an affine subspace of V, we know that there exists a vector $\mathbf{a} \in V$ and a linear subspace U' of V s.t. $U = \mathbf{a} + U'$. Moreover, by (ii), we have that $\mathbf{0} \in U$, and so by Theorem 5.1.1(b), we have that $U = \mathbf{0} + U'$.

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Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

1. (i) U is a linear subspace of V;
2. (ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then $\mathbf{0} \in U$ (by Theorem 3.1.7), and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

Suppose now that (ii) holds. Since U is an affine subspace of V, we know that there exists a vector $\mathbf{a} \in V$ and a linear subspace U' of V s.t. $U = \mathbf{a} + U'$. Moreover, by (ii), we have that $\mathbf{0} \in U$, and so by Theorem 5.1.1(b), we have that $U = \mathbf{0} + U'$. So, $U = U'$.

Corollary 5.1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

- (i) U is a linear subspace of V;
- (ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then $\mathbf{0} \in U$ (by Theorem 3.1.7), and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

Suppose now that (ii) holds. Since U is an affine subspace of V, we know that there exists a vector $\mathbf{a} \in V$ and a linear subspace U' of V s.t. $U = \mathbf{a} + U'$. Moreover, by (ii), we have that $\mathbf{0} \in U$, and so by Theorem 5.1.1(b), we have that $U = \mathbf{0} + U'$. So, $U = U'$. Since U' is a linear subspace of V, we see that (i) holds. $\square$

- Recall that the intersection of two linear subspaces is a linear subspace.

- Recall that the intersection of two linear subspaces is a linear subspace.
- In the case of affine subspaces, we have the following corollary.

Corollary 5.1.3

Let V be a vector space over a field $\mathbb{F}$, and let M_1 and M_2 be affine subspaces of V. Then either $M_1 \cap M_2 = \emptyset$, or $M_1 \cap M_2$ is an affine subspace of V.

Corollary 5.1.3

Let V be a vector space over a field $\mathbb{F}$, and let M_1 and M_2 be affine subspaces of V. Then either $M_1 \cap M_2 = \emptyset$, or $M_1 \cap M_2$ is an affine subspace of V.

Proof.

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Proof. WMA $M_1 \cap M_2 \neq \emptyset$, for otherwise we are done.

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Proof. WMA $M_1 \cap M_2 \neq \emptyset$, for otherwise we are done.

Fix any $\mathbf{a} \in M_1 \cap M_2$.

Corollary 5.1.3

Let V be a vector space over a field $\mathbb{F}$, and let M_1 and M_2 be affine subspaces of V. Then either $M_1 \cap M_2 = \emptyset$, or $M_1 \cap M_2$ is an affine subspace of V.

Proof. WMA $M_1 \cap M_2 \neq \emptyset$, for otherwise we are done.

Fix any $\mathbf{a} \in M_1 \cap M_2$. By Theorem 5.1.1, M_1 and M_2 can be written as $M_1 = \mathbf{a} + U_1$ and $M_2 = \mathbf{a} + U_2$, for some linear subspaces U_1 and U_2 of V.

Corollary 5.1.3

Let V be a vector space over a field $\mathbb{F}$, and let M_1 and M_2 be affine subspaces of V. Then either $M_1 \cap M_2 = \emptyset$, or $M_1 \cap M_2$ is an affine subspace of V.

Proof. WMA $M_1 \cap M_2 \neq \emptyset$, for otherwise we are done.

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2. Affine functions

Definition

Suppose that V_1 and V_2 are vector spaces over a field $\mathbb{F}$. A function $f : V_1 \to V_2$ is called an *affine function* if there exists a linear function $g : V_1 \to V_2$ and a vector $\mathbf{b} \in V_2$ s.t. for all $\mathbf{x} \in V_1$, we have that $f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{b}$.

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- Obviously, every linear function f is affine (we simply take $g := f$ and $\mathbf{b} := \mathbf{0}$).
- Moreover, we have the following proposition (next slide).

Proposition 5.2.1

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be an affine function. Then f is linear iff $f(\mathbf{0}) = \mathbf{0}$.

Proof.

Proposition 5.2.1

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be an affine function. Then f is linear iff $f(\mathbf{0}) = \mathbf{0}$.

Proof. If f is linear, then Proposition 4.1.6 guarantees that $f(\mathbf{0}) = \mathbf{0}$. For the reverse implication, we assume that $f(\mathbf{0}) = \mathbf{0}$, and we show that f is linear. Since f is an affine function, we know that there exists a linear function $g : V_1 \to V_2$ and a vector $\mathbf{b} \in V_2$ s.t. for all $\mathbf{x} \in V_1$, we have that $f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{b}$. But now

$$\mathbf{0} \;=\; f(\mathbf{0}) \;=\; g(\mathbf{0}) + \mathbf{b} \;\stackrel{(*)}{=}\; \mathbf{0} + \mathbf{b} \;=\; \mathbf{b}$$

where (*) follows from the fact that g is linear, and so $g(\mathbf{0}) = \mathbf{0}$ (by Proposition 4.1.6). So, $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in V_1$, that is, $f = g$. Since g is linear, so is f. $\square$

- We now state a number of results about affine functions.

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  - These results (and their proofs) are similar to various results that we proved for linear functions.
  - We omit the proofs here. However, all the proofs are in the Lecture Notes.

- Reminder:

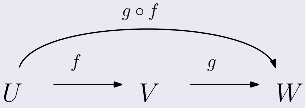
Theorem 4.1.7

Let U, V, and W be vector spaces over a field $\mathbb{F}$. Then all the following hold:

(a) for all linear functions $f, g : U \to V$, the function $f + g$ is linear;

(b) for all linear functions $f : U \to V$ and scalars $\alpha \in \mathbb{F}$, the function $\alpha f : U \to V$ is linear;

(c) for all linear functions $f : U \to V$ and $g : V \to W$, the function $g \circ f$ is liner.



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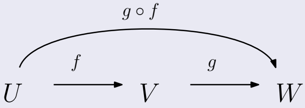
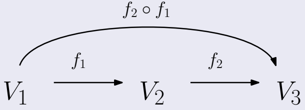


- For affine functions, we have the following analog of Theorem 4.1.7 (next slide).

Theorem 5.2.2

Let V_1, V_2, V_3 be vector spaces over a field $\mathbb{F}$. Then all the following hold:

a. for all affine functions $f_1, f_2 : V_1 \to V_2$, we have that $f_1 + f_2$ is an affine function;
b. for all affine functions $f : V_1 \to V_2$ and scalars α, we have that αf is an affine function;
c. for all affine functions $f_1 : V_1 \to V_2$ and $f_2 : V_2 \to V_3$, we have that $f_2 \circ f_1$ is an affine function.



- Proof: Lecture Notes.

- Reminder:

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

(b) $\text{Im}(f)$ is a subspace of V;

(c) for all subspaces V' of V, we have that $f^{-1}[V']$ is a subspace of U;

(d) $\text{Ker}(f)$ is a subspace of U.

- An analog of Theorem 4.2.3 for affine functions is split up over several theorems/corollaries, as follows.

Theorem 5.2.3

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, let $f : V_1 \to V_2$ be an affine function given by

$$f(\mathbf{x}) \;=\; g(\mathbf{x}) + \mathbf{b} \qquad \text{for all } \mathbf{x} \in V_1,$$

where $g : V_1 \to V_2$ is a linear function and $\mathbf{b}$ is a fixed vector in V_2, and let let $M_1 = \mathbf{a}_1 + U_1$ be an affine subspace of V_1 (where $\mathbf{a}_1$ is a vector and U_1 a linear subspace of V_1). Then

$$f[M_1] \;=\; (g(\mathbf{a}_1) + \mathbf{b}) + g[U_1],$$

and consequently, $f[M_1]$ is an affine subspace of V_2. Moreover,

$$\dim(f[M_1]) \;\leq\; \min\{\dim(M_1), \dim(V)\}.$$

- Proof: Lecture Notes.

Corollary 5.2.4

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be an affine function given by

$$f(\mathbf{x}) \;=\; g(\mathbf{x}) + \mathbf{b} \qquad \text{for all } \mathbf{x} \in V_1,$$

where $g : V_1 \to V_2$ is a linear function and $\mathbf{b}$ is a fixed vector in V_2. Then

$$\operatorname{Im}(f) \;=\; (g(\mathbf{a}_1) + \mathbf{b}) + \operatorname{Im}(g),$$

and consequently, $\operatorname{Im}(f)$ is an affine subspace of V_2. Moreover,

$$\dim(\operatorname{Im}(f)) \;=\; \operatorname{rank}(g) \;\leq\; \min\{\dim(V_1), \dim(V_2)\}.$$

- Proof: Lecture Notes.

- Geometric considerations:

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  - Suppose that $f : \mathbb{R}^m \to \mathbb{R}^n$ is an affine (possibly linear) function.
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  - Theorem 5.2.3 guarantees that f maps every affine subspace M of $\mathbb{R}^m$ onto an affine subspace of $\mathbb{R}^n$, and moreover, $\dim\big(f[M]\big) \leq \dim(M)$.

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  - Theorem 5.2.3 guarantees that f maps every affine subspace M of $\mathbb{R}^m$ onto an affine subspace of $\mathbb{R}^n$, and moreover, $\dim\big(f[M]\big) \leq \dim(M)$.
  - So, f maps lines onto lines or points, and it maps planes onto planes, lines, or points.
    - Obvious higher-dimensional generalizations apply.

Theorem 5.2.5

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be an affine function given by

$$f(\mathbf{x}) \quad = \quad g(\mathbf{x}) + \mathbf{b} \qquad \text{for all } \mathbf{x} \in V_1,$$

where $g : V_1 \to V_2$ is a linear function and $\mathbf{b}$ is a fixed vector in V_2. Further, let $M_2 = \mathbf{a}_2 + U_2$ be an affine subspace of V_2 (where $\mathbf{a}_2$ is a vector and U_2 a linear subspace of V_2). Then both the following hold:

(a) for all $\mathbf{a}_1 \in f^{-1}[M_2]$, we have that $f^{-1}[M_2] = \mathbf{a}_1 + g^{-1}[U_2]$;

(b) $f^{-1}[M_2]$ is either empty or an affine subspace of V_1.

- Proof: Lecture Notes.

Corollary 5.2.6

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be an affine function given by

$$f(\mathbf{x}) \;=\; g(\mathbf{x}) + \mathbf{b} \qquad \text{for all } \mathbf{x} \in V_1,$$

where $g : V_1 \to V_2$ is a linear function and $\mathbf{b}$ is a fixed vector in V_2. Further, let $\mathbf{c}$ be any vector in V_2. Then both the following hold:

(a) if $\mathbf{a} \in V_1$ is any solution of the equation $f(\mathbf{x}) = \mathbf{c}$,[a] then the solution set of the equation $f(\mathbf{x}) = \mathbf{c}$ is $\mathbf{a} + \text{Ker}(g)$;

(b) the solution set of the equation $f(\mathbf{x}) = \mathbf{c}$ is either empty or an affine subspace of V_1.

[a] This simply means that $f(\mathbf{a}) = \mathbf{c}$.

- Proof: Lecture Notes.

Corollary 5.2.7

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $\mathbf{b} \in \mathbb{F}^n$. Then both the following hold:

(a) if $\mathbf{a}$ is any solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{a} + \text{Nul}(A)$;

(b) if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, then its solution set is an affine subspace of $\mathbb{F}^m$.

- Proof: Lecture Notes.

Corollary 5.2.7

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $\mathbf{b} \in \mathbb{F}^n$. Then both the following hold:

(a) if $\mathbf{a}$ is any solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{a} + \text{Nul}(A)$;

(b) if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, then its solution set is an affine subspace of $\mathbb{F}^m$.

- Proof: Lecture Notes.

- Let's take a look at an example illustrating Corollary 5.2.7.

- Consider the the following matrix and vector, with entries understood to be in $\mathbb{Z}_3$:

$$A := \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 2 & 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} := \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

- Consider the the following matrix and vector, with entries understood to be in $\mathbb{Z}_3$:

$$A := \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 2 & 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} := \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

- Let us solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.

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$$A \;:=\; \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 2 & 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} \;:=\; \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

- Let us solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.
- We form the augmented matrix

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] \;=\; \left[\begin{array}{ccccc:c} 1 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \end{array}\right].$$

- By row reducing, we obtain

$$\text{RREF}\Big(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]\Big) = \left[\begin{array}{ccccc:c} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

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$$\text{RREF}\left(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]\right) = \left[\begin{array}{ccccc:c} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

- We see that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, and that the general solution of this equation is

$$\mathbf{x} = \begin{bmatrix} s+t+2 \\ 2s+2 \\ s+2t+1 \\ s \\ t \end{bmatrix}, \qquad \text{where } s, t \in \mathbb{Z}_3.$$

- Reminder:

$$\mathbf{x} = \begin{bmatrix} s+t+2 \\ 2s+2 \\ s+2t+1 \\ s \\ t \end{bmatrix}, \qquad \text{where } s, t \in \mathbb{Z}_3.$$

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- By separating parameters, we obtain

$$\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{where } s, t \in \mathbb{Z}_3.$$

- So, the solution set of the equation $A\mathbf{x} = \mathbf{b}$ is

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{Z}_3 \right\}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right).$$

- Reminder: the solution set of $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right).$$

- Reminder: the solution set of $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right).$$

- It is easy to check that $\mathbf{a} := \begin{bmatrix} 2 & 2 & 1 & 0 & 0 \end{bmatrix}^T$ is one solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, and that the null space of A is precisely

$$\mathrm{Nul}(A) \quad = \quad \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right).$$

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$$\text{Nul}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right).$$

- So, the solution set of $A\mathbf{x} = \mathbf{b}$ is precisely $\mathbf{a} + \text{Nul}(A)$, which is consistent with Corollary 5.2.7.

Corollary 5.2.7

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $\mathbf{b} \in \mathbb{F}^n$. Then both the following hold:

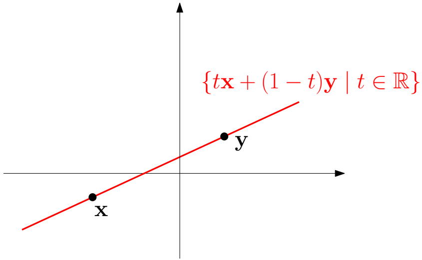
(a) if $\mathbf{a}$ is any solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{a} + \text{Nul}(A)$;

(b) if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, then its solution set is an affine subspace of $\mathbb{F}^m$.

- Geometric considerations:
  - Suppose that we are given a matrix $A \in \mathbb{R}^{n \times m}$ and a vector $\mathbf{b} \in \mathbb{R}^n$.
  - By Corollary 5.2.7(b), the solution set of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is either empty or an affine subspace of $\mathbb{R}^m$, i.e. a point (technically, a set that contains exactly one point), a line, a plane, or a higher-dimensional generalization in $\mathbb{R}^m$.

3. Affine combinations and affine hulls

- Recall from analytic geometry that if **x** and **y** are distinct points (vectors) in $\mathbb{R}^2$, then the line in $\mathbb{R}^2$ that passes through **x** and **y** is $\{t\mathbf{x} + (1-t)\mathbf{y} \mid t \in \mathbb{R}\}$.



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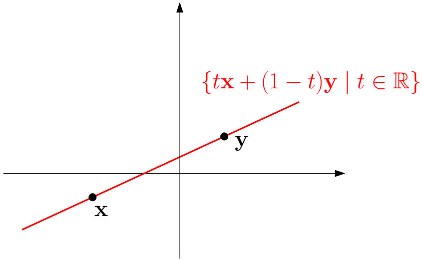

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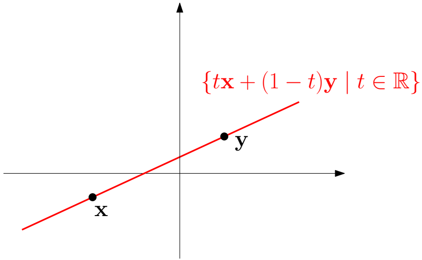

- This in fact holds for all distinct points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ (not just $\mathbb{R}^2$).
- Affine combinations are a generalization of this concept.

Definition

Suppose that $\mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 1$) are vectors in a vector space V over a field $\mathbb{F}$. An *affine combination* of $\mathbf{x}_1, \dots, \mathbf{x}_n$ is any sum of the form $\alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n$, where $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ satisfy $\alpha_1 + \dots + \alpha_n = 1$. The set of all affine combinations of $\mathbf{x}_1, \dots, \mathbf{x}_n$, denoted $\text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$, is called the *affine hull* (or *affine span*) of $\mathbf{x}_1, \dots, \mathbf{x}_n$. So, we have that

$$\text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n) \quad := \quad \Big\{ \sum_{i=1}^{n} \alpha_i \mathbf{x}_i \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, \ \sum_{i=1}^{n} \alpha_i = 1 \Big\}.$$

Definition

Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ ($n \geq 1$) are vectors in a vector space V over a field $\mathbb{F}$. An *affine combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is any sum of the form $\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ satisfy $\alpha_1 + \cdots + \alpha_n = 1$. The set of all affine combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, denoted $\text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, is called the *affine hull* (or *affine span*) of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. So, we have that

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- Since

$$\mathbf{x}_i \quad = \quad 0\mathbf{x}_1 + \cdots + 0\mathbf{x}_{i-1} + 1\mathbf{x}_i + 0\mathbf{x}_{i+1} + \cdots + 0\mathbf{x}_n$$

  for all $i \in \{1, \ldots, n\}$, we see that $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.

- Reminder:

Theorem 3.1.7

Let V be a vector space over a field $\mathbb{F}$, and let $U \subseteq V$. Then U is a subspace of V iff the following three conditions are satisfied:

(i) $\mathbf{0} \in U$;

(ii) U is closed under vector addition, that is, for all $\mathbf{u}, \mathbf{v} \in U$, we have that $\mathbf{u} + \mathbf{v} \in U$;

(iii) U is closed under scalar multiplication, that is, for all $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{u} \in U$.

Theorem 3.1.11

Let V be a vector space over a field $\mathbb{F}$, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$ $(k \geq 0)$. Then all the following hold:

(b) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of V;

Theorem 5.3.1

Let V be a vector space over a field $\mathbb{F}$, and let $M \subseteq V$. Then the following are equivalent:

1. M is an affine subspace of V;
2. M is **non-empty** and closed under affine combinations, that is, for all vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$ and $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$, we have that $\alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n \in M$.

Corollary 5.3.2

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in a vector space V over a field $\mathbb{F}$. Then $M := \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an affine subspace of V.

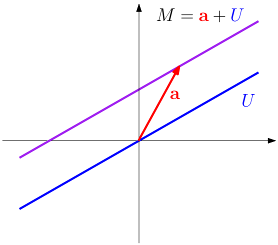
- Proof: Lecture Notes.

4. Affine frames and affine bases

4. Affine frames and affine bases

Definition

Let n be a non-negative integer, and let M be an n-dimensional affine subspace of a vector space V over a field $\mathbb{F}$. An *affine frame* of M is an ordered $(n+1)$-tuple $(\mathbf{a}, \mathbf{u}_1, \dots, \mathbf{u}_n)$ of vectors of V s.t. M can be written in the form $M = \mathbf{a} + U$, where U is a linear subspace of V, and $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis of U.



- By Theorem 3.2.7, if $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of a vector space V over a field $\mathbb{F}$, then every vector in V can be written as a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in a unique way.

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- Our next theorem is an analogue of this result for affine subspaces and affine frames.

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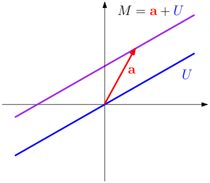
- Our next theorem is an analogue of this result for affine subspaces and affine frames.

Theorem 5.4.1

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $(\mathbf{a}, \mathbf{u}_1, \dots, \mathbf{u}_n)$ be an affine frame of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \mathbf{a} + \alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n$.

- Proof: Lecture Notes.



Definition

Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in a vector space V over a field $\mathbb{F}$, we say that vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$ are *affinely independent*, or that the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is *affinely independent*, if for all $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ s.t.

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n = \mathbf{0} \qquad \text{and} \qquad \alpha_1 + \cdots + \alpha_n = 0,$$

we have that $\alpha_1 = \cdots = \alpha_n = 0$.

Proposition 5.4.2

Let V be a vector space over a field $\mathbb{F}$, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 0$) be vectors in V. Then the following are equivalent:

1. (i) $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent;
2. (ii) there exists some $i \in \{0, 1, \dots, n\}$ s.t. vectors
$$\mathbf{x}_0 - \mathbf{x}_i, \dots, \mathbf{x}_{i-1} - \mathbf{x}_i, \mathbf{x}_{i+1} - \mathbf{x}_i, \dots, \mathbf{x}_n - \mathbf{x}_i$$
are linearly independent;
3. (iii) for all $i \in \{0, 1, \dots, n\}$, vectors
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are linearly independent.

Proof.

Proposition 5.4.2

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(ii) there exists some $i \in \{0, 1, \dots, n\}$ s.t. vectors

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are linearly independent;

(iii) for all $i \in \{0, 1, \dots, n\}$, vectors

$$\mathbf{x}_0 - \mathbf{x}_i, \dots, \mathbf{x}_{i-1} - \mathbf{x}_i, \mathbf{x}_{i+1} - \mathbf{x}_i, \dots, \mathbf{x}_n - \mathbf{x}_i$$

are linearly independent.

Proof. Obviously, (iii) implies (ii). We will show that (ii) implies (i), and that (i) implies (iii).

Proof (continued). Suppose that (ii) holds. Let us prove (i).

Proof (continued). Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA that $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

Proof (continued). Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

Fix scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n = \mathbf{0}$ and $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$. WTS $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$.

Proof (continued). Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

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Since $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$, we have that $\alpha_0 = -\alpha_1 - \dots - \alpha_n$, and so

$$\begin{aligned}
\mathbf{0} &= \alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \\
&= (-\alpha_1 - \dots - \alpha_n)\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \\
&= \alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0)
\end{aligned}$$

Proof (continued). Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

Fix scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n = \mathbf{0}$ and $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$. WTS $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$.

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\end{aligned}$$

Since vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent, we see that $\alpha_1 = \dots = \alpha_n = 0$.

Proof (continued). Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

Fix scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n = \mathbf{0}$ and $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$. WTS $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$.

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&= \alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0)
\end{aligned}$$

Since vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent, we see that $\alpha_1 = \dots = \alpha_n = 0$. Since $\alpha_0 = -\alpha_1 - \dots - \alpha_n$, it follows that $\alpha_0 = 0$. This proves (i).

Proof (continued). Suppose now that (i) holds. Let us prove (iii).

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By symmetry, it suffices to show that $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

Proof (continued). Suppose now that (i) holds. Let us prove (iii).

By symmetry, it suffices to show that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Fix scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0) = \mathbf{0}$.

Proof (continued). Suppose now that (i) holds. Let us prove (iii).

By symmetry, it suffices to show that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Fix scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0) = \mathbf{0}$. Then

$$(\underbrace{-\alpha_1 - \dots - \alpha_n}_{:=\alpha_0})\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \quad = \quad \mathbf{0}.$$

Proof (continued). Suppose now that (i) holds. Let us prove (iii).

By symmetry, it suffices to show that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Fix scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0) = \mathbf{0}$. Then

$$\underbrace{(-\alpha_1 - \dots - \alpha_n)}_{:=\alpha_0}\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n = \mathbf{0}.$$

Since $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent, we now get that $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$, and we deduce that (iii) holds. $\square$

Definition

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$. An *affine basis* (also called a *barycentric frame*) of M is a non-empty ordered set $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of vectors in M s.t.

- vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ are affinely independent;
- $M = \mathrm{Aff}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$.

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- $M = \text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$.

Theorem 5.4.3

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ be an affine basis of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$, called the *barycentric coordinates* of $\mathbf{x}$ with respect to the affine basis $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$, s.t. $\mathbf{x} = \sum_{i=0}^{n} \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^{n} \alpha_i = 1$.

- Proof: Lecture Notes.

Theorem 5.4.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

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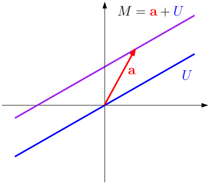
- **Remark:** Since every affine frame of an n-dimensional affine subspace contains $n+1$ vectors, Theorem 5.4.4 implies that every affine basis of an n-dimensional affine subspace contains exactly $n+1$ vectors.



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Proof.

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2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof. First, we know that (i) and (ii) are, respectively, equivalent to (1) and (2) below:

1. vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are **affinely** independent and $M = \text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are **linearly** independent and $M = \mathbf{x}_0 + \text{Span}(\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$.

So, it suffices to show that (1) and (2) are equivalent.

Theorem 5.4.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;

(ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof (continued). WTS the following are equivalent:

1. vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are **affinely** independent and $M = \text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are **linearly** independent and $M = \mathbf{x}_0 + \text{Span}(\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$.

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Proof (continued). WTS the following are equivalent:

1. vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are **affinely** independent and $M = \text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are **linearly** independent and $M = \mathbf{x}_0 + \text{Span}(\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$.

By Proposition 5.4.2, vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent iff vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

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Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

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Proof (continued). WTS the following are equivalent:

1. vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are **affinely** independent and $M = \text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are **linearly** independent and $M = \mathbf{x}_0 + \text{Span}(\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$.

By Proposition 5.4.2, vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent iff vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

It now remains to show that
$\text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{x}_0 + \text{Span}(\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$.

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Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

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Proof (continued). WTS the following are equivalent:

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For this, we compute (next slide):

Theorem 5.4.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;
2. $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof (continued).

$$\begin{aligned}
&\mathrm{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n) \\
= \;& \left\{\alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n \mid \alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F},\ \alpha_0 + \alpha_1 + \cdots + \alpha_n = 1\right\} \\
= \;& \left\{(1 - \alpha_1 - \cdots - \alpha_n)\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}\right\} \\
= \;& \left\{\mathbf{x}_0 + \alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \cdots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0) \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}\right\} \\
= \;& \mathbf{x}_0 + \mathrm{Span}(\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0).
\end{aligned}$$

This completes the argument. □

Theorem 5.4.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an affine basis of M;

(ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

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1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

- **Remark:** If $M = \{\mathbf{a}\}$ is a one-element affine subspace of a vector space V over a field $\mathbb{F}$, then $(\mathbf{a})$ is the (unique) affine frame and $\{\mathbf{a}\}$ the (unique) affine basis of M.

- **Remark:** Suppose we are given a matrix $A \in \mathbb{F}^{n \times m}$ and a vector $\mathbf{b} \in \mathbb{F}^n$ (where $\mathbb{F}$ is a field). By Corollary 5.2.7, the solution set of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is either empty or an affine subspace of $\mathbb{F}^m$. Moreover, we have the following:

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    - if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is **inconsistent**, then its solution set is **empty**, and consequently, it is **not** and affine subspace of $\mathbb{F}^m$ and therefore does **not** have an affine frame or an affine basis;

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    - if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is **inconsistent**, then its solution set is **empty**, and consequently, it is **not** and affine subspace of $\mathbb{F}^m$ and therefore does **not** have an affine frame or an affine basis;
    - if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, say $\mathbf{x}_0$, then $\{\mathbf{x}_0\}$ is the solution set of $A\mathbf{x} = \mathbf{b}$, and we see that $(\mathbf{x}_0)$ is the (unique) affine frame and $\{\mathbf{x}_0\}$ the (unique) affine basis of the solution set of $A\mathbf{x} = \mathbf{b}$;

- **Remark:** Suppose we are given a matrix $A \in \mathbb{F}^{n \times m}$ and a vector $\mathbf{b} \in \mathbb{F}^n$ (where $\mathbb{F}$ is a field). By Corollary 5.2.7, the solution set of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is either empty or an affine subspace of $\mathbb{F}^m$. Moreover, we have the following:
  - if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is **inconsistent**, then its solution set is **empty**, and consequently, it is **not** and affine subspace of $\mathbb{F}^m$ and therefore does **not** have an affine frame or an affine basis;
  - if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, say $\mathbf{x}_0$, then $\{\mathbf{x}_0\}$ is the solution set of $A\mathbf{x} = \mathbf{b}$, and we see that $(\mathbf{x}_0)$ is the (unique) affine frame and $\{\mathbf{x}_0\}$ the (unique) affine basis of the solution set of $A\mathbf{x} = \mathbf{b}$;
  - if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has more than one solution, then an affine frame and an affine basis of the solution set of $A\mathbf{x} = \mathbf{b}$ can be computed by following the procedure from the solution of Example 5.4.5 (next slide).

Example 5.4.5

Consider the following matrix and vector, both with entries in $\mathbb{Z}_2$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Show that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, and consequently (by Corollary 5.2.7(b)), an affine subspace of $\mathbb{Z}_2^6$. Find an affine frame and an affine basis of the solution set of $A\mathbf{x} = \mathbf{b}$.

Solution.

Example 5.4.5

Consider the following matrix and vector, both with entries in $\mathbb{Z}_2$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Show that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, and consequently (by Corollary 5.2.7(b)), an affine subspace of $\mathbb{Z}_2^6$. Find an affine frame and an affine basis of the solution set of $A\mathbf{x} = \mathbf{b}$.

Solution. We form the augmented matrix

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right],$$

and by row reducing, we obtain (next slide):

Solution (continued).

$$\text{RREF}\Big(\left[\begin{array}{c:c} A & \mathbf{b} \end{array} \right] \Big) = \left[\begin{array}{cccccc:c} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Solution (continued).

$$\mathrm{RREF}\Big(\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] \Big) = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So, the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{bmatrix} r+s+t+1 \\ s+t+1 \\ r+t+1 \\ r \\ s \\ t \end{bmatrix} \qquad \text{where } r, s, t \in \mathbb{Z}_2.$$

Solution (continued).

$$\text{RREF}\Big(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]\Big) = \left[\begin{array}{cccccc:c} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So, the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{bmatrix} r+s+t+1 \\ s+t+1 \\ r+t+1 \\ r \\ s \\ t \end{bmatrix} \qquad \text{where } r, s, t \in \mathbb{Z}_2.$$

In particular, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, and so by Corollary 5.2.7(b), the solution set of this equation is an affine subspace of $\mathbb{Z}_2^6$.

Solution (continued). So, the solution set of $A\mathbf{x} = \mathbf{b}$ is:

$$S := \left\{ \begin{bmatrix} r+s+t+1 \\ s+t+1 \\ r+t+1 \\ r \\ s \\ t \end{bmatrix} \mid r,s,t \in \mathbb{Z}_2 \right\}$$

$$= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid r,s,t \in \mathbb{Z}_2 \right\}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

Solution (continued). Reminder: The solution set of $A\mathbf{x} = \mathbf{b}$ is:

$$S = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

Solution (continued). Reminder: The solution set of $A\mathbf{x} = \mathbf{b}$ is:

$$S = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

We now see that

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

is an affine frame of the solution set S of $A\mathbf{x} = \mathbf{b}$, whereas (by Theorem 5.4.4):

Solution (continued).

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is an affine basis of S. $\square$