# Linear Algebra 1

# Lecture #11

# Linear functions (part II)

Irena Penev

December 20-21, 2023

- This lecture has four parts:

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    1. The effect of a linear function on linearly independent and spanning sets

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    4. Isomorphisms

1. The effect of a linear function on linearly independent and spanning sets

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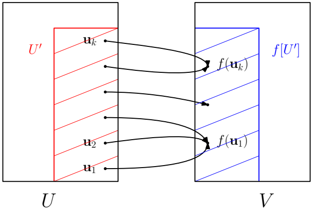
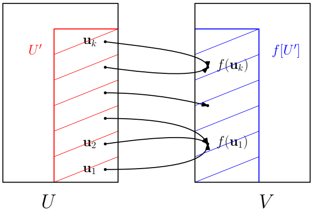
### Theorem 4.2.11

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, and set $U' := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then all the following hold:

a. $U'$ is a subspace of $U$, and $f[U']$ is a subspace of $V$;
b. $f[U'] = f\left[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\right] = \text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big)$, i.e. vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $f[U'] = f\left[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\right]$;
c. $\dim\big(f[U']\big) \leq \dim(U') \leq k$.



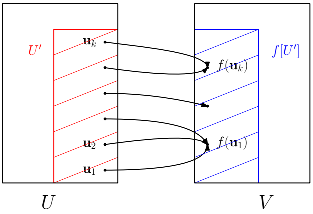



**Theorem 4.2.11**

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, and set $U' := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then all the following hold:

(a) $U'$ is a subspace of $U$, and $f[U']$ is a subspace of $V$;

*Proof of (a).*

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(a) $U'$ is a subspace of $U$, and $f[U']$ is a subspace of $V$;

*Proof of (a).* The fact that $U'$ is a subspace of $U$ follows immediately from Theorem 3.1.11, and the fact that $f[U']$ is a subspace of $V$ follows from 4.2.3(a). This proves (a).

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(b) $f[U'] = f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big] = \text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big)$, i.e. vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $f[U'] = f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big]$;

*Proof of (b).*

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*Proof of (b).*

$$\begin{aligned}
\text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big) &= \big\{\alpha_1 f(\mathbf{u}_1) + \dots + \alpha_k f(\mathbf{u}_k) \mid \alpha_1, \dots, \alpha_k \in \mathbb{F}\big\} \\
&\overset{(*)}{=} \Big\{f\big(\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k\big) \mid \alpha_1, \dots, \alpha_k \in \mathbb{F}\Big\} \\
&\overset{(**)}{=} \big\{f(\mathbf{u}) \mid \mathbf{u} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big\} \\
&= f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big] \;=\; f[U'],
\end{aligned}$$

where (*) follows from the linearity of the $f$ (and more precisely, from Prop. 4.1.5), and (**) follows from the definition of span.

## Theorem 4.2.11

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \rightarrow V$ be a linear function. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, and set $U' := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then all the following hold:

(a) $U'$ is a subspace of $U$, and $f[U']$ is a subspace of $V$;

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(c) $\dim\big(f[U']\big) \leq \dim(U') \leq k$.

*Proof of (c).*

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- (c) $\dim\big(f[U']\big) \leq \dim(U') \leq k$.

*Proof of (c).* By hypothesis, $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a spanning set of $U'$.

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*Proof of (c).* By hypothesis, $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a spanning set of $U'$. So, by Theorem 3.2.14, some subset of that spanning set, say $\{\mathbf{u}_{i_1}, \dots, \mathbf{u}_{i_m}\}$ (with $1 \leq i_1 < \dots < i_m \leq k$) is a basis of $U'$.

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Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, and set $U' := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then all the following hold:

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*Proof of (c).* By hypothesis, $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a spanning set of $U'$. So, by Theorem 3.2.14, some subset of that spanning set, say $\{\mathbf{u}_{i_1}, \dots, \mathbf{u}_{i_m}\}$ (with $1 \leq i_1 < \dots < i_m \leq k$) is a basis of $U'$. So, $\dim(U') = m \leq k$. But now $\{\mathbf{u}_{i_1}, \dots, \mathbf{u}_{i_m}\}$ is a spanning set of $U'$. So, by part (b) applied to the set $\{\mathbf{u}_{i_1}, \dots, \mathbf{u}_{i_m}\}$, we get that $\{f(\mathbf{u}_{i_1}), \dots, f(\mathbf{u}_{i_m})\}$ is a spanning set of $f[U']$. We now apply Theorem 3.2.14 again, and we deduce that some subset of $\{f(\mathbf{u}_{i_1}), \dots, f(\mathbf{u}_{i_m})\}$ is a basis of $f[U']$, and so $\dim\big(f[U']\big) \leq m$. $\square$

## Theorem 4.2.11

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, and set $U' := \mathrm{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then all the following hold:

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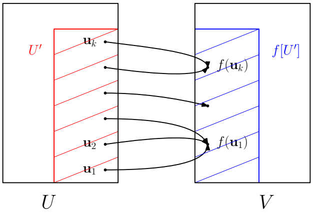
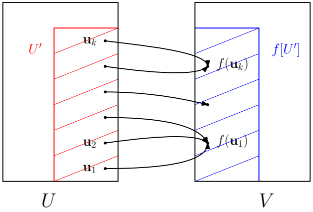
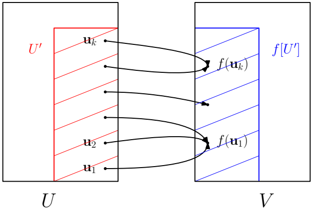





## Corollary 4.2.12

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, let $f : U \to V$ be a linear function, and let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a spanning set of $U$. Then $\text{Im}(f) = \text{Span}(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k))$ and $\text{rank}(f) = \dim\Big(\text{Span}(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k))\Big) \leq k$.

*Proof.*

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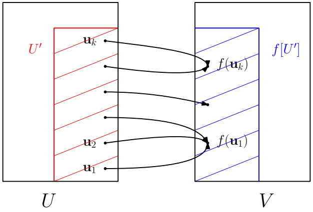
*Proof.* By hypothesis, $U = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$.

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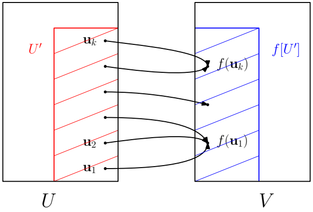
*Proof.* By hypothesis, $U = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. So, by Theorem 4.2.11(b), we have that
$\text{Im}(f) = f[U] = \text{Span}(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k))$,

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*Proof.* By hypothesis, $U = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. So, by Theorem 4.2.11(b), we have that $\text{Im}(f) = f[U] = \text{Span}(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k))$, and by Theorem 4.2.11(c), we have that $\text{rank}(f) = \dim(\text{Im}(f)) = \dim(f[U]) \leq k$. $\square$

## Theorem 4.2.13

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, let $f : U \to V$ be a linear function, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$. Then all the following hold:

(a) if $f$ is one-to-one and vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly independent in $U$, then vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ are linearly independent in $V$;

(b) if vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ are linearly independent in $V$, then vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly independent in $U$;

(c) if $f$ is onto and vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ span $U$, then vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $V$;

(d) if $f$ is one-to-one and vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $V$, then vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ span $U$.

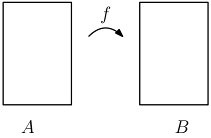
- Proof: Lecture Notes.
- Informal summary: next slide.

- Theorem 4.2.13 (schematically and informally):

$$f : U \xrightarrow{\text{linear}} V$$

| | | | |
|---|---|---|---|
| (a)-(b) | $\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly independent | $\xRightarrow{\text{if } f \text{ is 1-1}}$ <br> $\xLeftarrow[\text{always}]{}$ | $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ are linearly independent |
| (c)-(d) | $\mathbf{u}_1, \dots, \mathbf{u}_k$ span $U$ | $\xRightarrow{\text{if } f \text{ is onto}}$ <br> $\xLeftarrow[\text{if } f \text{ is 1-1}]{}$ | $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $V$ |

- **Dimension considerations:**




  - As we know, for any function $f : A \to B$, where $A$ and $B$ are finite sets, the following hold:
    - if $f$ is one-to-one, then $|A| \leq |B|$;
    - if $f$ is onto, then $|A| \geq |B|$;
    - if $f$ is a bijection, then $|A| = |B|$.

    (Actually, the above is true even if we allow $A$ and $B$ to be infinite, but to make sense of the statement, we would need infinite cardinals. We omit the details.)

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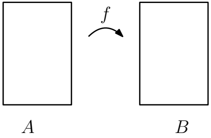


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    (Actually, the above is true even if we allow $A$ and $B$ to be infinite, but to make sense of the statement, we would need infinite cardinals. We omit the details.)
  - In the case of **linear** functions, Theorem 4.2.14 (next slide) gives us a very similar statement, only involving dimension (rather than cardinality) of the domain and codomain.

**Theorem 4.2.14**

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

a. if $f$ is one-to-one, then $\dim(U) \leq \dim(V)$;
b. if $f$ is onto, then $\dim(U) \geq \dim(V)$;
c. if $f$ is an isomorphism, then $\dim(U) = \dim(V)$.

*Proof.*

## Theorem 4.2.14

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

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(c) if $f$ is an isomorphism, then $\dim(U) = \dim(V)$.

*Proof.* Obviously, (a) and (b) together imply (c).

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*Proof.* Obviously, (a) and (b) together imply (c). So, it is enough to prove (a) and (b).

## Theorem 4.2.14

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) if $f$ is one-to-one, then $\dim(U) \leq \dim(V)$;

*Proof (continued).* (a) We prove the contrapositive: we assume that $\dim(U) > \dim(V)$ (and in particular, $\dim(V)$ is finite), and we prove that $f$ is **not** one-to-one.

### Theorem 4.2.14

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

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Set $n := \dim(V)$.

## Theorem 4.2.14

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Set $n := \dim(V)$. Since $\dim(U) > \dim(V)$, we know that $U$ has a linearly independent set of size greater than $n$.

- Indeed, if $U$ is finite-dimensional, then any one of its bases is a linearly independent set of size $\dim(U) > n$, and if $U$ is infinite-dimensional, then Proposition 3.2.18 guarantees that $U$ has linearly independent sets of any finite size.

So, fix a linearly independent set $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of $U$, with $k > n$.

## Theorem 4.2.14

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

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So, fix a linearly independent set $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of $U$, with $k > n$. Since $\dim(V) = n$, Theorem 3.2.17(a) guarantees that the set $\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\}$ is linearly dependent. But now Theorem 4.2.13(a) guarantees that $f$ is not one-to-one.

## Theorem 4.2.14

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(b) if $f$ is onto, then $\dim(U) \geq \dim(V)$;

*Proof (continued).* (b) Assume that $f$ is onto; we must show that $\dim(U) \geq \dim(V)$.

**Theorem 4.2.14**

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(b) if $f$ is onto, then $\dim(U) \geq \dim(V)$;

*Proof (continued).* (b) Assume that $f$ is onto; we must show that $\dim(U) \geq \dim(V)$. We may assume that $n := \dim(U)$ is finite, for otherwise, we are done.

**Theorem 4.2.14**

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

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*Proof (continued).* (b) Assume that $f$ is onto; we must show that $\dim(U) \geq \dim(V)$. We may assume that $n := \dim(U)$ is finite, for otherwise, we are done. We must show that $\dim(V) \leq n$.

## Theorem 4.2.14

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

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*Proof (continued).* (b) Assume that $f$ is onto; we must show that $\dim(U) \geq \dim(V)$. We may assume that $n := \dim(U)$ is finite, for otherwise, we are done. We must show that $\dim(V) \leq n$.

Fix any basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of $U$.

**Theorem 4.2.14**

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

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*Proof (continued).* (b) Assume that $f$ is onto; we must show that $\dim(U) \geq \dim(V)$. We may assume that $n := \dim(U)$ is finite, for otherwise, we are done. We must show that $\dim(V) \leq n$.

Fix any basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of $U$. In particular, vectos $\mathbf{u}_1, \dots, \mathbf{u}_n$ span $U$, and so since $f$ is onto, Theorem 4.2.13(c) guarantees that vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_n)$ span $V$.

## Theorem 4.2.14

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(b) if $f$ is onto, then $\dim(U) \geq \dim(V)$;

*Proof (continued).* (b) Assume that $f$ is onto; we must show that $\dim(U) \geq \dim(V)$. We may assume that $n := \dim(U)$ is finite, for otherwise, we are done. We must show that $\dim(V) \leq n$.

Fix any basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of $U$. In particular, vectos $\mathbf{u}_1, \dots, \mathbf{u}_n$ span $U$, and so since $f$ is onto, Theorem 4.2.13(c) guarantees that vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_n)$ span $V$. But then by Theorem 3.2.14, some subset of $\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_n)\}$ is a basis of $V$, and it follows that $\dim(V) \leq n$. □

**Theorem 4.2.14**

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

a. if $f$ is one-to-one, then $\dim(U) \leq \dim(V)$;
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c. if $f$ is an isomorphism, then $\dim(U) = \dim(V)$.

2. Computing bases of the images and preimages of subspaces under linear functions

**Proposition 4.2.15**

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of $f$, let $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}^m$ $(k \geq 1)$, and set $U := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then

$$f[U] \;=\; \text{Col}\Big(A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big),$$

and moreover, the pivot columns of the matrix $A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}$ form a basis of $f[U]$.

- First an example, then a proof.

## Example 4.2.16

Let $f : \mathbb{Z}_2^5 \to \mathbb{Z}_2^4$ be the linear function whose standard matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix},$$

and consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{Z}_2^5$. Set $U := \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$. Find a basis for $f[U]$.

*Solution.*

*Solution.* Our goal is to find the pivot columns of the matrix $A\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix}$, since by Proposition 4.2.15, those columns form a basis of $f[U]$. First, by multiplying matrices, we obtain

$$A\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Solution (continued).* By row reducing, we obtain

$$\text{RREF}\Big(A\left[\begin{array}{cccc}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4\end{array}\right]\Big) = \left[\begin{array}{cccc}1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

*Solution (continued).* By row reducing, we obtain

$$\text{RREF}\Big(A\left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]\Big) = \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

As we can see, the pivot columns of $A\left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]$ are its first and fourth column.

*Solution (continued).* By row reducing, we obtain

$$\operatorname{RREF}\Big(A\left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]\Big) = \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

As we can see, the pivot columns of $A\left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]$ are its first and fourth column. Therefore,

$$\left\{\left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \end{array}\right]\right\}$$

is a basis of $f[U]$. $\square$

**Proposition 4.2.15**

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of $f$, let $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}^m$ $(k \geq 1)$, and set $U := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then

$$f[U] \quad = \quad \text{Col}\Big(A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big),$$

and moreover, the pivot columns of the matrix $A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}$ form a basis of $f[U]$.

*Proof.*

## Proposition 4.2.15

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of $f$, let $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}^m$ $(k \geq 1)$, and set $U := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then

$$f[U] \;=\; \text{Col}\Big(A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big),$$

and moreover, the pivot columns of the matrix $A \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}$ form a basis of $f[U]$.

*Proof.* First, we compute (next slide):

*Proof (continued).*

$$\begin{aligned} f[U] &= f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big] \\ &\overset{(*)}{=} \text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big) \\ &\overset{(**)}{=} \text{Col}\Big(\begin{bmatrix} f(\mathbf{u}_1) & \dots & f(\mathbf{u}_k) \end{bmatrix}\Big) \\ &\overset{(***)}{=} \text{Col}\Big(\begin{bmatrix} A\mathbf{u}_1 & \dots & A\mathbf{u}_k \end{bmatrix}\Big) \\ &\overset{(****)}{=} \text{Col}\Big(A\begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big), \end{aligned}$$

where (*) follows from Theorem 4.2.11(b), (**) follows from the definition of the column space, and (***) follows from the fact that $A$ is the standard matrix of $f$, and (****) follows from the definition of matrix multiplication.

*Proof (continued).*

$$\begin{aligned}
f[U] \quad &= \quad f\big[\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)\big] \\
&\overset{(*)}{=} \quad \text{Span}\big(f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\big) \\
&\overset{(**)}{=} \quad \text{Col}\Big(\begin{bmatrix} f(\mathbf{u}_1) & \dots & f(\mathbf{u}_k) \end{bmatrix}\Big) \\
&\overset{(***)}{=} \quad \text{Col}\Big(\begin{bmatrix} A\mathbf{u}_1 & \dots & A\mathbf{u}_k \end{bmatrix}\Big) \\
&\overset{(****)}{=} \quad \text{Col}\Big(A\begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}\Big),
\end{aligned}$$

where (*) follows from Theorem 4.2.11(b), (**) follows from the definition of the column space, and (***) follows from the fact that $A$ is the standard matrix of $f$, and (****) follows from the definition of matrix multiplication. By Theorem 3.3.4, the pivot columns of a matrix form a basis of the column space of that matrix, and the result follows. □

## Proposition 4.2.18

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n\times m}$ be the standard matrix of $f$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}^n$ $(k \geq 1)$, and set $V := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ Then

$$\begin{aligned} f^{-1}[V] &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c|cccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] = \mathbf{0} \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] \in \text{Nul}\left( \left[\begin{array}{c|cccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \right) \right\}. \end{aligned}$$

- First an example, then a proof.

## Example 4.2.19

Consider the linear function $f : \mathbb{R}^4 \to \mathbb{R}^5$ whose standard matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -4 & 0 \\ -2 & -3 & -6 & 1 \\ 4 & 0 & 0 & 0 \\ 2 & -1 & -2 & 0 \end{bmatrix},$$

and consider the following vectors in $\mathbb{R}^5$:

- $\mathbf{v}_1 = \begin{bmatrix} -1 & 6 & 9 & -4 & 1 \end{bmatrix}^T$;
- $\mathbf{v}_2 = \begin{bmatrix} 2 & 2 & -2 & 8 & 5 \end{bmatrix}$;
- $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}^T$;
- $\mathbf{v}_4 = \begin{bmatrix} 0 & -2 & -3 & -1 & -1 \end{bmatrix}^T$;
- $\mathbf{v}_5 = \begin{bmatrix} 0 & -1 & -2 & 1 & 0 \end{bmatrix}^T$;
- $\mathbf{v}_6 = \begin{bmatrix} -3 & -1 & 2 & -11 & -6 \end{bmatrix}^T$.

Set $V := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_6)$. Find a basis of $f^{-1}[V]$.

*Solution.* We apply Proposition 4.2.18.

*Solution.* We apply Proposition 4.2.18. We first form the matrix

$$C := \left[\begin{array}{c:cccccc} A & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right]$$

$$= \left[\begin{array}{rrrr:rrrrrr} 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -3 \\ 0 & -2 & -4 & 0 & 6 & 2 & 0 & -2 & -1 & -1 \\ -2 & -3 & -6 & 1 & 9 & -2 & 0 & -3 & -2 & 2 \\ 4 & 0 & 0 & 0 & -4 & 8 & -1 & -1 & 1 & -11 \\ 2 & -1 & -2 & 0 & 1 & 5 & 0 & -1 & 0 & -6 \end{array}\right],$$

and we find the general solution of the matrix-vector equation

$$\underbrace{\left[\begin{array}{c:cccccc} A & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right]}_{=C} \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] = \mathbf{0},$$

where the vector $\mathbf{x}$ has four entries (because $A$ has four columns) and the vector $\mathbf{y}$ has six entries (because we have six vectors $\mathbf{v}_1, \ldots, \mathbf{v}_6$).

*Solution (continued).* By row reducing, we obtain

$$\mathrm{RREF}(C) \quad = \quad \left[\begin{array}{cccc:cccccc} 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 & -3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

*Solution (continued).* By row reducing, we obtain

$$\mathrm{RREF}(C) = \left[\begin{array}{cccc|cccccc} 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 & -3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

So, the general solution of our matrix-vector equation is

$$\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] = \left[\begin{array}{c} q - 2r + 3t \\ -2p + 3q + r - s \\ p \\ 2q + r + 2t \\ \hline q \\ r \\ -s \\ s \\ -t \\ t \end{array}\right], \qquad \text{where } p, q, r, s, t \in \mathbb{R}.$$

*Solution (continued).* By row reducing, we obtain

$$\mathrm{RREF}(C) = \left[\begin{array}{cccc:cccccc} 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 & -3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

So, the general solution of our matrix-vector equation is

$$\left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] = \left[\begin{array}{c} q - 2r + 3t \\ -2p + 3q + r - s \\ p \\ 2q + r + 2t \\ \hdashline q \\ r \\ -s \\ s \\ -t \\ t \end{array}\right], \qquad \text{where } p, q, r, s, t \in \mathbb{R}.$$

But as per Proposition 4.2.18, we only need **x**!

*Solution.* So, we simply ignore the part below the horizontal dotted line, and we obtain:

$$\mathbf{x} \quad = \quad \begin{bmatrix} q - 2r + 3t \\ -2p + 3q + r - s \\ p \\ 2q + r + 2t \end{bmatrix}, \qquad \text{where } p, q, r, s, t \in \mathbb{R}.$$

*Solution.* So, we simply ignore the part below the horizontal dotted line, and we obtain:

$$\mathbf{x} = \begin{bmatrix} q-2r+3t \\ -2p+3q+r-s \\ p \\ 2q+r+2t \end{bmatrix}, \qquad \text{where } p,q,r,s,t \in \mathbb{R}.$$

By separating parameters, we obtain

$$\mathbf{x} = p\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + q\begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix} + r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix},$$

where $p,q,r,s,t \in \mathbb{R}$.

*Solution.* So, we simply ignore the part below the horizontal dotted line, and we obtain:

$$\mathbf{x} \quad = \quad \begin{bmatrix} q-2r+3t \\ -2p+3q+r-s \\ p \\ 2q+r+2t \end{bmatrix}, \qquad \text{where } p,q,r,s,t \in \mathbb{R}.$$

By separating parameters, we obtain

$$\mathbf{x} \quad = \quad p\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + q\begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix} + r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix},$$

where $p,q,r,s,t \in \mathbb{R}$.

In view of Proposition 4.2.18, we now have that (next slide):

*Solution.*

$$
\begin{aligned}
f^{-1}[V] &= \Big\{ p \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix} \mid \\
&\qquad \mid p, q, r, s, t \in \mathbb{R} \Big\} \\
&= \operatorname{Span}\Big( \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix} \Big) \\
&= \operatorname{Col}\Big( \underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B} \Big).
\end{aligned}
$$

*Solution (continued).* Reminder:

$$f^{-1}[V] \quad = \quad \mathrm{Col}\Bigg( \underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B} \Bigg).$$

*Solution (continued).* Reminder:

$$f^{-1}[V] \quad = \quad \text{Col}\Bigg(\underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B}\Bigg).$$

We note that the five vectors that we obtained in the second-to-last line above are not necessarily linearly independent, and so to find an actual basis of $f^{-1}[V]$, we row reduce the matrix $B$ and use Theorem 3.3.4. Indeed, Theorem 3.3.4 guarantees that the pivot columns of $B$ form a basis of $\text{Col}(B) = f^{-1}[V]$.

- In fact, we can immediately see that they are not linearly independent: no five vectors in $\mathbb{R}^4$ are linearly independent (by Theorem 3.2.17(a)).
- More generally, though, the reason our computation does not necessarily yield linearly independent vectors is because we "cut off" the entries below the vertical dotted line.

*Solution (continued).* Reminder:

$$f^{-1}[V] \quad = \quad \mathrm{Col}\Bigg(\underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B}\Bigg).$$

*Solution (continued).* Reminder:

$$f^{-1}[V] \quad = \quad \mathrm{Col}\Bigg( \underbrace{\begin{bmatrix} 0 & 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}}_{=:B} \Bigg).$$

By row reducing, we obtain

$$\mathrm{RREF}(B) \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 7/5 \\ 0 & 0 & 1 & 0 & -4/5 \\ 0 & 0 & 0 & 1 & 17/5 \end{bmatrix}.$$

Thus, the pivot columns of $B$ are its leftmost four columns, and those four columns form a basis of $f^{-1}[V]$.

*Solution (continued).* So, our final answer is that

$$\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis of $f^{-1}[V]$. $\square$

## Proposition 4.2.18

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of $f$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}^n$ $(k \geq 1)$, and set $V := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ Then

$$\begin{aligned} f^{-1}[V] &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c:ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] = \mathbf{0} \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] \in \text{Nul}\Big( \left[\begin{array}{c:ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \Big) \right\}. \end{aligned}$$

*Proof.*

## Proposition 4.2.18

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n\times m}$ be the standard matrix of $f$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}^n$ $(k \geq 1)$, and set $V := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ Then

$$
\begin{aligned}
f^{-1}[V] &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c:ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] = \mathbf{0} \right\} \\
&= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] \in \text{Nul}\Big( \left[\begin{array}{c:ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \Big) \right\}.
\end{aligned}
$$

*Proof.* Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array}\right]$.

## Proposition 4.2.18

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, let $A \in \mathbb{F}^{n\times m}$ be the standard matrix of $f$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}^n$ $(k \geq 1)$, and set $V := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ Then

$$\begin{aligned} f^{-1}[V] &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c:ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] = \mathbf{0} \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{F}^m \mid \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{y} \end{array}\right] \in \text{Nul}\Big( \left[\begin{array}{c:ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \Big) \right\}. \end{aligned}$$

*Proof.* Set $A = \left[ \begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$. Then for all vectors $\mathbf{x} = \left[ \begin{array}{ccc} x_1 & \dots & x_m \end{array} \right]^T$ in $\mathbb{F}^m$, we have the following sequence of equivalent statements (next slide):

*Proof (continued).*

$$
\begin{aligned}
& \mathbf{x} \in f^{-1}[V] \\
\iff \quad & f(\mathbf{x}) \in \underbrace{\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)}_{=V} \\
\overset{(*)}{\iff} \quad & A\mathbf{x} \in \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \\
\iff \quad & \underbrace{x_1\mathbf{a}_1 + \dots + x_m\mathbf{a}_m}_{=A\mathbf{x}} \in \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \\
\overset{(**)}{\iff} \quad & \exists \alpha_1, \dots, \alpha_k \in \mathbb{F} \text{ s.t. } x_1\mathbf{a}_1 + \dots + x_m\mathbf{a}_m = \alpha_1\mathbf{v}_1 + \dots + \alpha_k\mathbf{v}_k \\
\iff \quad & \exists \alpha_1, \dots, \alpha_k \in \mathbb{F} \text{ s.t. } x_1\mathbf{a}_1 + \dots + x_m\mathbf{a}_m - \alpha_1\mathbf{v}_1 - \dots - \alpha_k\mathbf{v}_k = \mathbf{0},
\end{aligned}
$$

where (*) follows from the fact that $A$ is the standard matrix of $f$, and (**) follows from the definition of span.

*Proof (continued).*

$$\mathbf{x} \in f^{-1}[V]$$

$$\overset{\text{previous slide}}{\iff} \quad \exists \alpha_1, \dots, \alpha_k \in \mathbb{F} \text{ s.t. } x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m - \alpha_1\mathbf{v}_1 - \cdots - \alpha_k\mathbf{v}_k = \mathbf{0}$$

$$\overset{(***)}{\iff} \quad \exists y_1, \dots, y_k \in \mathbb{F} \text{ s.t. } x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m + y_1\mathbf{v}_1 + \cdots + y_k\mathbf{v}_k = \mathbf{0}$$

$$\iff \quad \exists y_1, \dots, y_k \in \mathbb{F} \text{ s.t. } \left[\begin{array}{ccc|ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} x_1 \\ \vdots \\ x_m \\ \hline y_1 \\ \vdots \\ y_k \end{array}\right] = \mathbf{0}$$

$$\iff \quad \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] = \mathbf{0},$$

where (***) follows via substitution $y_i := -\alpha_i \ \forall i \in \{1, \dots, k\}$.

*Proof (continued).*

$$
\begin{aligned}
& \mathbf{x} \in f^{-1}[V] \\
\overset{\text{previous slide}}{\iff} \quad & \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] = \mathbf{0} \\
\overset{(****)}{\iff} \quad & \exists \mathbf{y} \in \mathbb{F}^k \text{ s.t. } \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] \in \mathrm{Nul}\Big( \left[\begin{array}{c|ccc} A & \mathbf{v}_1 & \dots & \mathbf{v}_k \end{array}\right] \Big),
\end{aligned}
$$

where (****) follows from the definition of the null space. The result is now immediate. □

3. Linear functions and bases

3. Linear functions and bases

- Reminder:

**Theorem 1.10.5**

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be any vectors in $\mathbb{F}^n$. Then there exists a **unique** linear function $f : \mathbb{F}^m \to \mathbb{F}^n$ that satisfies $f(\mathbf{e}_1) = \mathbf{a}_1, \dots, f(\mathbf{e}_m) = \mathbf{a}_m$, where $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of $\mathbb{F}^m$. Moreover, this linear function $f$ is given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^m$, where $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$.

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- Our next goal is to generalize Theorem 1.10.5 to linear functions $f : U \to V$, where $U$ and $V$ are vector spaces over a field $\mathbb{F}$, and $U$ is **finite-dimensional**.
  - Instead of using the standard basis $\mathcal{E}_m = \{\mathbf{e}_1, \dots, \mathbf{e}_m\}$, we will use an arbitrary basis of $U$.

- Suppose that $V$ is a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $V$.

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- By Theorem 3.2.7, every vector of $V$ can be written as linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a unique way, that is, $\forall \mathbf{v} \in V\ \exists! \alpha_1, \ldots, \alpha_n \in \mathbb{F}$ s.t.

$$\mathbf{v} \quad := \quad \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n,$$

and the *coordinate vector* of $\mathbf{v}$ with respect to the basis $\mathcal{B}$ is defined to be

$$\left[\ \mathbf{v}\ \right]_{\mathcal{B}} \quad := \quad \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

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$$\mathbf{v} \quad := \quad \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n,$$

  and the *coordinate vector* of $\mathbf{v}$ with respect to the basis $\mathcal{B}$ is defined to be

$$\left[\ \mathbf{v}\ \right]_{\mathcal{B}} \quad := \quad \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

- As our next proposition shows, $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.
  - It essentially allows us to "translate" vectors of an $n$-dimensional vector space ($n \neq 0$) into vectors in $\mathbb{F}^n$.

### Proposition 4.3.1

Let $V$ be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $V$. Then $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

*Proof.*

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*Proof.* We start by proving that $\left[\ \cdot\ \right]_{\mathcal{B}}$ is linear.

1. Fix $\mathbf{x}, \mathbf{y} \in V$. WTS $\left[\ \mathbf{x} + \mathbf{y}\ \right]_{\mathcal{B}} = \left[\ \mathbf{x}\ \right]_{\mathcal{B}} + \left[\ \mathbf{y}\ \right]_{\mathcal{B}}$.

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*Proof.* We start by proving that $\left[\ \cdot\ \right]_{\mathcal{B}}$ is linear.

1. Fix $\mathbf{x}, \mathbf{y} \in V$. WTS $\left[\ \mathbf{x}+\mathbf{y}\ \right]_{\mathcal{B}} = \left[\ \mathbf{x}\ \right]_{\mathcal{B}} + \left[\ \mathbf{y}\ \right]_{\mathcal{B}}$. Set $\left[\ \mathbf{x}\ \right]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$ and $\left[\ \mathbf{y}\ \right]_{\mathcal{B}} = \begin{bmatrix} \beta_1 & \dots & \beta_n \end{bmatrix}^T$. Then $\mathbf{x} = \alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n$ and $\mathbf{y} = \beta_1\mathbf{v}_1 + \dots + \beta_n\mathbf{v}_n$; consequently,

$$\mathbf{x}+\mathbf{y} \ = \ (\alpha_1+\beta_1)\mathbf{v}_1 + \dots + (\alpha_n+\beta_n)\mathbf{v}_n,$$

and so $\left[\ \mathbf{x}+\mathbf{y}\ \right]_{\mathcal{B}} = \begin{bmatrix} \alpha_1+\beta_1 & \dots & \alpha_n+\beta_n \end{bmatrix}^T$.

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Let $V$ be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of $V$. Then $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

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$$\mathbf{x} + \mathbf{y} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \dots + (\alpha_n + \beta_n)\mathbf{v}_n,$$

and so $\left[\ \mathbf{x} + \mathbf{y}\ \right]_{\mathcal{B}} = \left[\ \alpha_1 + \beta_1 \ \ \dots \ \ \alpha_n + \beta_n\ \right]^T$. We now have that (next slide):

## Proposition 4.3.1

Let $V$ be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of $V$. Then $\left[ \cdot \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

*Proof (continued).*

$$\begin{aligned}
\left[ \mathbf{x} + \mathbf{y} \right]_{\mathcal{B}} &= \begin{bmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_n + \beta_n \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \\
&= \left[ \mathbf{x} \right]_{\mathcal{B}} + \left[ \mathbf{y} \right]_{\mathcal{B}}.
\end{aligned}$$

**Proposition 4.3.1**

Let $V$ be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of $V$. Then $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

*Proof (continued).* Similarly (details: Lecture Notes):

2. $\forall \mathbf{x} \in V$, $\alpha \in \mathbb{F}$: $\left[\ \alpha\mathbf{x}\ \right]_{\mathcal{B}} = \alpha \left[\ \mathbf{x}\ \right]_{\mathcal{B}}$.

**Proposition 4.3.1**

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So, $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is linear.

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*Proof (continued).* It remains to show that $\left[\ \cdot\ \right]_{\mathcal{B}}$ is a bijection, i.e. that it is one-to-one and onto $\mathbb{F}^n$.

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Since $V$ and $\mathbb{F}^n$ are both $n$ dimensional, Corollary 4.2.10 guarantees that $f$ is one-to-one iff $f$ is onto $\mathbb{F}^n$.

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Since $V$ and $\mathbb{F}^n$ are both $n$ dimensional, Corollary 4.2.10 guarantees that $f$ is one-to-one iff $f$ is onto $\mathbb{F}^n$. So, it is enough to show that $f$ is onto $\mathbb{F}^n$.

Fix $\begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T \in \mathbb{F}^n$.

## Proposition 4.3.1

Let $V$ be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of $V$. Then $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

*Proof (continued).* It remains to show that $\left[\ \cdot\ \right]_{\mathcal{B}}$ is a bijection, i.e. that it is one-to-one and onto $\mathbb{F}^n$.

Since $V$ and $\mathbb{F}^n$ are both $n$ dimensional, Corollary 4.2.10 guarantees that $f$ is one-to-one iff $f$ is onto $\mathbb{F}^n$. So, it is enough to show that $f$ is onto $\mathbb{F}^n$.

Fix $\begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T \in \mathbb{F}^n$. Set $\mathbf{v} := \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$.

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Let $V$ be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of $V$. Then $\left[\ \cdot\ \right]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

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- Reminder:

**Theorem 1.10.5**

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be any vectors in $\mathbb{F}^n$. Then there exists a **unique** linear function $f : \mathbb{F}^m \to \mathbb{F}^n$ that satisfies $f(\mathbf{e}_1) = \mathbf{a}_1, \dots, f(\mathbf{e}_m) = \mathbf{a}_m$, where $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of $\mathbb{F}^m$. Moreover, this linear function $f$ is given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^m$, where $A = \left[ \begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$.

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- Let's generalize this!

## Theorem 4.3.2

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and assume that $U$ is finite-dimensional. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis of $U$, and let $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$.[a] Then there exists a unique linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, if the vector space $U$ is non-trivial (i.e. $n \neq 0$), then this unique linear function $f : U \to V$ satisfies the following: for all $\mathbf{u} \in U$, we have that

$$f(\mathbf{u}) \quad = \quad \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n,$$

where $\left[ \begin{array}{c} \mathbf{u} \end{array} \right]_{\mathcal{B}} = \left[ \begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array} \right]^T$. On the other hand, if $U$ is trivial (i.e. $U = \{\mathbf{0}\}$),[b] then $f : U \to V$ is given by $f(\mathbf{0}) = \mathbf{0}$.

[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_n$ are arbitrary vectors in $V$. They are not necessarily pairwise distinct.

[b] Note that in this case, we have that $n = 0$ and $\mathcal{B} = \emptyset$.

*Proof.*

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*Proof.* Suppose first that the vector space $U$ is trivial, i.e. $n = 0$ and $U = \{\mathbf{0}\}$. Then the function $f : U \rightarrow V$ given by $f(\mathbf{0}) = \mathbf{0}$ is obviously linear, and moreover, it vacuously satisfies $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$ (because $n = 0$, and so both $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are empty lists of vectors). The uniqueness of $f$ follows from Proposition 4.1.6.

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From now on, we assume that the vector space $U$ is non-trivial, i.e. that $n \neq 0$.

*Proof.* Suppose first that the vector space $U$ is trivial, i.e. $n = 0$ and $U = \{\mathbf{0}\}$. Then the function $f : U \to V$ given by $f(\mathbf{0}) = \mathbf{0}$ is obviously linear, and moreover, it vacuously satisfies $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$ (because $n = 0$, and so both $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are empty lists of vectors). The uniqueness of $f$ follows from Proposition 4.1.6.

From now on, we assume that the vector space $U$ is non-trivial, i.e. that $n \neq 0$. We must prove the existence and the uniqueness of the linear function $f$ satisfying the required properties.

*Proof (continued).* **Existence.** Let $f : U \to V$ be defined as in the statement of the theorem, i.e. for all $\mathbf{u} \in U$, we set

$$f(\mathbf{u}) = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n,$$

where $\left[ \begin{array}{c} \mathbf{u} \end{array} \right]_{\mathcal{B}} = \left[ \begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array} \right]^T$.

*Proof (continued).* **Existence.** Let $f : U \to V$ be defined as in the statement of the theorem, i.e. for all $\mathbf{u} \in U$, we set

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where $\begin{bmatrix} \mathbf{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$. Note that this means that for all $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, we have that

$$f(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n) = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n.$$

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$$f(\alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n) \;=\; \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n.$$

Let us show that $f$ is linear and satisfies
$f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$.

*Proof (continued).* **Existence.** Let $f : U \to V$ be defined as in the statement of the theorem, i.e. for all $\mathbf{u} \in U$, we set

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where $\begin{bmatrix} \mathbf{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$. Note that this means that for all $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, we have that

$$f(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n) = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n.$$

Let us show that $f$ is linear and satisfies $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. For the latter, we note that for all $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned} f(\mathbf{u}_i) &= f(0\mathbf{u}_1 + \cdots + 0\mathbf{u}_{i-1} + 1\mathbf{u}_i + 0\mathbf{u}_{i+1} + \cdots + 0\mathbf{u}_n) \\ &= 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{i-1} + 1\mathbf{v}_i + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_n \\ &= \mathbf{v}_i. \end{aligned}$$

This proves that $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$.

*Proof (continued).* Let us now show that $f$ is linear. We verify that $f$ satisfies the two axioms from the definition of a linear function.

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1. Fix $\mathbf{x}, \mathbf{y} \in U$. WTS $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$.

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1. Fix $\mathbf{x}, \mathbf{y} \in U$. WTS $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$. Set $\left[\ \mathbf{x}\ \right]_{\mathcal{B}} = \left[\ \alpha_1 \quad \dots \quad \alpha_n\ \right]^T$ and $\mathbf{y} = \left[\ \beta_1 \quad \dots \quad \beta_n\ \right]^T$.

*Proof (continued).* Let us now show that $f$ is linear. We verify that $f$ satisfies the two axioms from the definition of a linear function.

1. Fix $\mathbf{x}, \mathbf{y} \in U$. WTS $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$. Set $\left[\begin{array}{c} \mathbf{x} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array}\right]^T$ and $\mathbf{y} = \left[\begin{array}{ccc} \beta_1 & \dots & \beta_n \end{array}\right]^T$. We then have that $\left[\begin{array}{c} \mathbf{x}+\mathbf{y} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1+\beta_1 & \dots & \alpha_n+\beta_n \end{array}\right]^T$,

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$$\begin{array}{rcl} f(\mathbf{x}+\mathbf{y}) & \stackrel{(*)}{=} & (\alpha_1+\beta_1)\mathbf{v}_1 + \dots + (\alpha_n+\beta_n)\mathbf{v}_n \\ & = & (\alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n) + (\beta_1\mathbf{v}_1 + \dots + \beta_n\mathbf{v}_n) \\ & \stackrel{(**)}{=} & f(\mathbf{x}) + f(\mathbf{y}), \end{array}$$

where both (*) and (**) follow from the construction of $f$.

*Proof (continued).* 2. Fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$. WTS $f(\alpha \mathbf{u}) = \alpha f(\mathbf{u})$.

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*Proof (continued).* 2. Fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$. WTS $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$.
Set $\left[\begin{array}{c} \mathbf{u} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array}\right]^T$. Then
$\left[\begin{array}{c} \alpha\mathbf{u} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha\alpha_1 & \dots & \alpha\alpha_n \end{array}\right]^T$,

*Proof (continued).* 2. Fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$. WTS $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$. Set $\left[\ \mathbf{u}\ \right]_{\mathcal{B}} = \left[\ \alpha_1 \quad \dots \quad \alpha_n\ \right]^T$. Then $\left[\ \alpha\mathbf{u}\ \right]_{\mathcal{B}} = \left[\ \alpha\alpha_1 \quad \dots \quad \alpha\alpha_n\ \right]^T$, and we see that

$$\begin{aligned} f(\alpha\mathbf{u}) &\overset{(*)}{=} (\alpha\alpha_1)\mathbf{v}_1 + \cdots + (\alpha\alpha_n)\mathbf{v}_n \\ &= \alpha(\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n) \\ &\overset{(**)}{=} \alpha f(\mathbf{u}), \end{aligned}$$

where both (*) and (**) follow from the construction of $f$.

*Proof (continued).* 2. Fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$. WTS $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$. Set $\left[\begin{array}{c} \mathbf{u} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array}\right]^T$. Then $\left[\begin{array}{c} \alpha\mathbf{u} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha\alpha_1 & \dots & \alpha\alpha_n \end{array}\right]^T$, and we see that

$$\begin{array}{rcl} f(\alpha\mathbf{u}) & \stackrel{(*)}{=} & (\alpha\alpha_1)\mathbf{v}_1 + \dots + (\alpha\alpha_n)\mathbf{v}_n \\ & = & \alpha(\alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n) \\ & \stackrel{(**)}{=} & \alpha f(\mathbf{u}), \end{array}$$

where both (*) and (**) follow from the construction of $f$.

By 1. and 2., we see that $f$ is linear. This completes the proof of existence.

*Proof (continued).* **Uniqueness.** Let $f_1, f_2 : U \to V$ be linear functions that satisfy $f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n$ and $f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n$. WTS $f_1 = f_2$.

*Proof (continued).* **Uniqueness.** Let $f_1, f_2 : U \to V$ be linear functions that satisfy $f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n$ and $f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n$. WTS $f_1 = f_2$. Fix $\mathbf{u} \in U$. WTS

$f_1(\mathbf{u}) = f_2(\mathbf{u})$.

*Proof (continued).* **Uniqueness.** Let $f_1, f_2 : U \to V$ be linear functions that satisfy $f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n$ and $f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n$. WTS $f_1 = f_2$. Fix $\mathbf{u} \in U$. WTS $f_1(\mathbf{u}) = f_2(\mathbf{u})$. Set $\left[\begin{array}{c} \mathbf{u} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array}\right]^T$.

*Proof (continued).* **Uniqueness.** Let $f_1, f_2 : U \to V$ be linear functions that satisfy $f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n$ and $f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n$. WTS $f_1 = f_2$. Fix $\mathbf{u} \in U$. WTS $f_1(\mathbf{u}) = f_2(\mathbf{u})$. Set $\left[\begin{array}{c} \mathbf{u} \end{array}\right]_{\mathcal{B}} = \left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_n \end{array}\right]^T$. Then

| | | | |
|---|---|---|---|
| $f_1(\mathbf{u})$ | $=$ | $f_1(\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n)$ | |
| | | | by the linearity of $f_1$ (and more precisely, by Proposition 4.1.5) |
| | $=$ | $\alpha_1 f_1(\mathbf{u}_1) + \dots + \alpha_n f_1(\mathbf{u}_n)$ | |
| | | | because $f_1(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_1(\mathbf{u}_n) = \mathbf{v}_n$ |
| | $=$ | $\alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n$ | |
| | | | because $f_2(\mathbf{u}_1) = \mathbf{v}_1, \dots, f_2(\mathbf{u}_n) = \mathbf{v}_n$ |
| | $=$ | $\alpha_1 f_2(\mathbf{u}_1) + \dots + \alpha_n f_2(\mathbf{u}_n)$ | |
| | | | by the linearity of $f_2$ (and more precisely, by Proposition 4.1.5) |
| | $=$ | $f_2(\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n)$ | |
| | $=$ | $f_2(\mathbf{u})$. | |

Thus, $f_1 = f_2$. This proves uniqueness. □

## Theorem 4.3.2

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and assume that $U$ is finite-dimensional. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis of $U$, and let $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$.[a] Then there exists a unique linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, if the vector space $U$ is non-trivial (i.e. $n \neq 0$), then this unique linear function $f : U \to V$ satisfies the following: for all $\mathbf{u} \in U$, we have that

$$f(\mathbf{u}) \quad = \quad \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n,$$

where $\left[\ \mathbf{u}\ \right]_{\mathcal{B}} = \left[\ \alpha_1 \quad \dots \quad \alpha_n\ \right]^T$. On the other hand, if $U$ is trivial (i.e. $U = \{\mathbf{0}\}$),[b] then $f : U \to V$ is given by $f(\mathbf{0}) = \mathbf{0}$.

[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_n$ are arbitrary vectors in $V$. They are not necessarily pairwise distinct.

[b] Note that in this case, we have that $n = 0$ and $\mathcal{B} = \emptyset$.

## Corollary 4.3.3

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and assume that $U$ is finite-dimensional. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set of vectors in $U$, and let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$.[a] Then there exists a linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_k) = \mathbf{v}_k$. Moreover, if $V$ is non-trivial, then this linear function $f$ is unique iff $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of $U$.

[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_k$ are arbitrary vectors in $V$. They are not necessarily pairwise distinct.

- **Remark:** If $V$ is trivial (i.e. $V = \{\mathbf{0}\}$, and consequently $\mathbf{v}_1 = \dots = \mathbf{v}_k = \mathbf{0}$), then there exists exactly one **function** from $U$ to $V$, this function maps all elements of $U$ to $\mathbf{0}$, and obviously, it is linear.

## Corollary 4.3.3

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and assume that $U$ is finite-dimensional. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set of vectors in $U$, and let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$.[a] Then there exists a linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_k) = \mathbf{v}_k$. Moreover, if $V$ is non-trivial, then this linear function $f$ is unique iff $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of $U$.

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[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_k$ are arbitrary vectors in $V$. They are not necessarily pairwise distinct.

*Proof (outline).*

## Corollary 4.3.3

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and assume that $U$ is finite-dimensional. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set of vectors in $U$, and let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$.[a] Then there exists a linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_k) = \mathbf{v}_k$. Moreover, if $V$ is non-trivial, then this linear function $f$ is unique iff $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of $U$.

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[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_k$ are arbitrary vectors in $V$. They are not necessarily pairwise distinct.

*Proof (outline).* Using Theorem 3.2.19, we extend $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ to a basis of $U$, and then we apply Theorem 4.3.2. The details are left as an exercise. $\square$

4. Isomorphisms

4. Isomorphisms

- Recall that, for vector spaces $U$ and $V$ over a field $\mathbb{F}$, a function $f : U \to V$ is an *isomorphism* if it is linear and a bijection.

4. Isomorphisms

- Recall that, for vector spaces $U$ and $V$ over a field $\mathbb{F}$, a function $f : U \to V$ is an *isomorphism* if it is linear and a bijection.
- Vector spaces $U$ and $V$ (over the same field $\mathbb{F}$) are *isomorphic*, and we write $U \cong V$, if there exits an isomorphism $f : U \to V$.

## Proposition 4.4.1

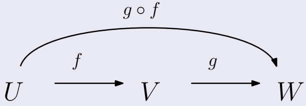
Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be an isomorphism. Then $f^{-1} : V \to U$ is also an isomorphism.

$$U \underset{f^{-1}}{\overset{f}{\rightleftarrows}} V$$

*Proof.* The same as for isomorphisms $f : \mathbb{F}^n \to \mathbb{F}^n$ (details: Lecture Notes). $\square$

## Proposition 4.4.2

Let $U$, $V$, and $W$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ and $g : V \to W$ be isomorphisms. Then $g \circ f : U \to W$ is an isomorphism.




*Proof.*

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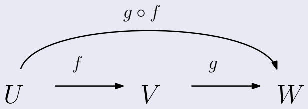


*Proof.* Since $f : U \to V$ and $g : V \to W$ are linear functions (because they are isomorphisms), Proposition 4.1.7 guarantees that their composition $g \circ f : U \to W$ is also linear.

## Proposition 4.4.2

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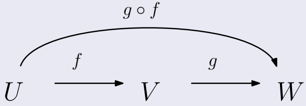


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Since $f : U \to V$ and $g : V \to W$ are bijections, Proposition 1.10.17 guarantees that $g \circ f : U \to W$ is also a bijection.

## Proposition 4.4.2

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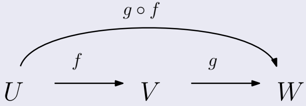


*Proof.* Since $f : U \to V$ and $g : V \to W$ are linear functions (because they are isomorphisms), Proposition 4.1.7 guarantees that their composition $g \circ f : U \to W$ is also linear.

Since $f : U \to V$ and $g : V \to W$ are bijections, Proposition 1.10.17 guarantees that $g \circ f : U \to W$ is also a bijection.

So, $g \circ f : U \to W$ is linear and a bijection, i.e. it is an isomorphism. $\square$

## Theorem 4.4.3

Let $U$, $V$, and $W$ be vector spaces over a field $\mathbb{F}$. Then all the following hold:

(a) $U \cong U$;

(b) if $U \cong V$, then $V \cong U$;

(c) if $U \cong V$ and $V \cong W$, then $U \cong W$.

*Proof.*

### Theorem 4.4.3

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*Proof.* (a) Clearly, $\text{Id}_U : U \to U$ (the identity function on $U$) is an isomorphism. So, $U \cong U$.

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*Proof.* (a) Clearly, $\text{Id}_U : U \to U$ (the identity function on $U$) is an isomorphism. So, $U \cong U$.

(b) Suppose that $U \cong V$. Then there exists an isomorphism $f : U \to V$.

**Theorem 4.4.3**

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(c) Suppose that $U \cong V$ and $V \cong W$. Then there exist isomorphisms $f : U \to V$ and $g : V \to W$.

## Theorem 4.4.3

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- Reminder: Theorem 4.2.13 (schematically and informally):

$$f : U \overset{\text{linear}}{\longrightarrow} V$$

| | | | |
|---|---|---|---|
| (a)-(b) | $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent | $\overset{\text{if } f \text{ is 1-1}}{\Longrightarrow}$ / $\underset{\text{always}}{\Longleftarrow}$ | $f(\mathbf{u}_1), \ldots, f(\mathbf{u}_k)$ are linearly independent |
| (c)-(d) | $\mathbf{u}_1, \ldots, \mathbf{u}_k$ span $U$ | $\overset{\text{if } f \text{ is onto}}{\Longrightarrow}$ / $\underset{\text{if } f \text{ is 1-1}}{\Longleftarrow}$ | $f(\mathbf{u}_1), \ldots, f(\mathbf{u}_k)$ span $V$ |

- Reminder: Theorem 4.2.13 (schematically and informally):

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## Theorem 4.4.4

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, let $f : U \to V$ be an isomorphism, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$. Then all the following hold:

(a) vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly independent in $U$ iff vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ are linearly independent in $V$;

(b) vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ span $U$ iff vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $V$;

(c) $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of $U$ iff $\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\}$ is a basis of $V$.

*Proof.* This follows from Theorem 4.2.13 (details: Lecture Notes).

**Theorem 4.4.4**

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, let $f : U \to V$ be an isomorphism, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$. Then all the following hold:

(a) vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly independent in $U$ iff vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ are linearly independent in $V$;

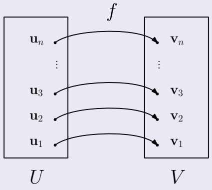
(b) vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ span $U$ iff vectors $f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)$ span $V$;

(c) $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of $U$ iff $\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_k)\}$ is a basis of $V$.

- Proposition 4.4.5 (next slide) is a converse of sorts of Theorem 4.4.4(c).
  - It essentially states that any linear function that (injectively) maps a basis onto a basis is an isomorphism.

## Proposition 4.4.5

Let $U$ and $V$ be finite-dimensional vector spaces over a field $\mathbb{F}$. Assume that $\dim(U) = \dim(V) =: n$. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for $U$, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $V$. Then there exists a unique linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, this linear function $f$ is an isomorphism.




*Proof.*

## Proposition 4.4.5

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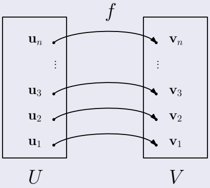


*Proof.* The existence and uniqueness of the linear function $f$ follows from Theorem 4.3.2.

## Proposition 4.4.5

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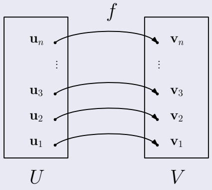
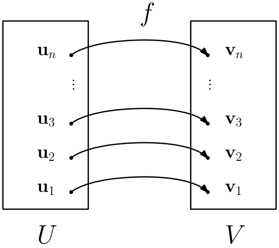
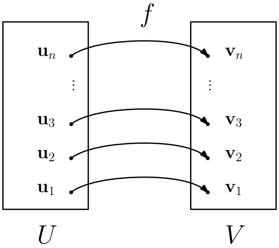


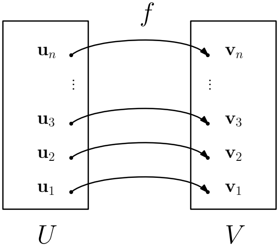

*Proof.* The existence and uniqueness of the linear function $f$ follows from Theorem 4.3.2. But by hypothesis, $U$ and $V$ are finite-dimensional vector spaces satisfying $\dim(U) = \dim(V)$, and so by Corollary 4.2.10, it is enough to show that $f$ is onto.




*Proof (continued).* Fix $\mathbf{v} \in V$.




*Proof (continued).* Fix $\mathbf{v} \in V$. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $V$, we know that there exist scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$.



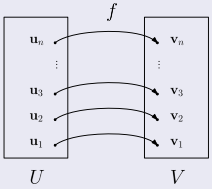

*Proof (continued).* Fix $\mathbf{v} \in V$. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $V$, we know that there exist scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{v} = \alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n$. But now

$$\begin{aligned} f(\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n) &\overset{(*)}{=} \alpha_1 f(\mathbf{u}_1) + \dots + \alpha_n f(\mathbf{u}_n) \\ &= \alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n \\ &= \mathbf{v}, \end{aligned}$$

where (*) follows from the linearity of $f$ (and more precisely, from Proposition 4.1.5). So, $f$ is onto, and we are done. □

## Proposition 4.4.5

Let $U$ and $V$ be finite-dimensional vector spaces over a field $\mathbb{F}$. Assume that $\dim(U) = \dim(V) =: n$. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for $U$, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $V$. Then there exists a unique linear function $f : U \to V$ s.t. $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, this linear function $f$ is an isomorphism.



- Reminder:

**Theorem 4.2.14**

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) if $f$ is one-to-one, then $\dim(U) \leq \dim(V)$;

(b) if $f$ is onto, then $\dim(U) \geq \dim(V)$;

(c) if $f$ is an isomorphism, then $\dim(U) = \dim(V)$.

- Reminder:

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- By Theorem 4.2.14(c), any two isomorphic vector spaces have the same dimension.

- Reminder:

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(c) if $f$ is an isomorphism, then $\dim(U) = \dim(V)$.

- By Theorem 4.2.14(c), any two isomorphic vector spaces have the same dimension.
- Theorem 4.4.6 (next slide) guarantees that, in the case of **finite-dimensional** vector spaces, the converse is also true: any two vector spaces (over the same field) that have the same finite dimension are isomorphic.
  - We give two proofs of Theorem 4.4.6!

**Theorem 4.4.6**

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

- **Warning:** This theorem is only true for finite-dimensional vector spaces, and it becomes false for infinite-dimensional ones.

### Theorem 4.4.6

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

*Proof#1.*

## Theorem 4.4.6

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

*Proof#1.* If $U$ and $V$ are isomorphic, then Theorem 4.2.14(c) guarantees that $\dim(U) = \dim(V)$.

## Theorem 4.4.6

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

*Proof#1.* If $U$ and $V$ are isomorphic, then Theorem 4.2.14(c) guarantees that $\dim(U) = \dim(V)$.

Suppose, conversely, that $\dim(U) = \dim(V) =: n$.

## Theorem 4.4.6

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*Proof#1.* If $U$ and $V$ are isomorphic, then Theorem 4.2.14(c) guarantees that $\dim(U) = \dim(V)$.

Suppose, conversely, that $\dim(U) = \dim(V) =: n$. Fix any basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of $U$ and any basis $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ of $V$.

**Theorem 4.4.6**

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

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Suppose, conversely, that $\dim(U) = \dim(V) =: n$. Fix any basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of $U$ and any basis $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ of $V$. By Proposition 4.3.1, $\left[\ \cdot\ \right]_{\mathcal{B}} : U \to \mathbb{F}^n$ and $\left[\ \cdot\ \right]_{\mathcal{C}} : V \to \mathbb{F}^n$ are both isomorphisms,

### Theorem 4.4.6

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

*Proof#1.* If $U$ and $V$ are isomorphic, then Theorem 4.2.14(c) guarantees that $\dim(U) = \dim(V)$.

Suppose, conversely, that $\dim(U) = \dim(V) =: n$. Fix any basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of $U$ and any basis $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ of $V$. By Proposition 4.3.1, $\left[\ \cdot\ \right]_{\mathcal{B}} : U \to \mathbb{F}^n$ and $\left[\ \cdot\ \right]_{\mathcal{C}} : V \to \mathbb{F}^n$ are both isomorphisms, and consequently, $U \cong \mathbb{F}^n$ and $V \cong \mathbb{F}^n$.

**Theorem 4.4.6**

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

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Suppose, conversely, that $\dim(U) = \dim(V) =: n$. Fix any basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of $U$ and any basis $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ of $V$. By Proposition 4.3.1, $\left[\ \cdot\ \right]_{\mathcal{B}} : U \to \mathbb{F}^n$ and $\left[\ \cdot\ \right]_{\mathcal{C}} : V \to \mathbb{F}^n$ are both isomorphisms, and consequently, $U \cong \mathbb{F}^n$ and $V \cong \mathbb{F}^n$. But now Theorem 4.4.3 guarantees that $U \cong V$. □

Theorem 4.4.6

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

*Proof#2.*

## Theorem 4.4.6

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

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## Theorem 4.4.6

Let $U$ and $V$ be **finite-dimensional** vector spaces over a field $\mathbb{F}$. Then $U$ and $V$ are isomorphic iff $\dim(U) = \dim(V)$.

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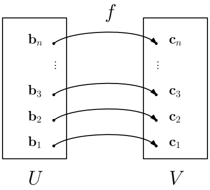

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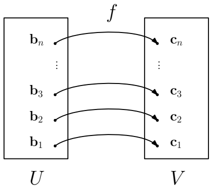



So, $U$ and $V$ are isomorphic. $\square$

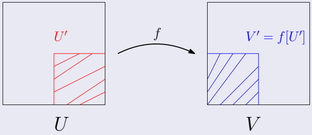
## Proposition 4.4.7

Let $U$ and $V$ be a vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be an isomorphism, and let $U' \subseteq U$. Then $U'$ is a subspace of $U$ iff $V' := f[U']$ is a subspace of $V$. Moreover, in this case, all the following hold:

(a) the function $f' : U' \to V'$ given by $f'(\mathbf{u}) = f(\mathbf{u})$ for all $\mathbf{u} \in U'$ is an isormophism;

(b) $U' \cong V'$;

(c) $\dim(U') = \dim(V')$.




*Proof.* Lecture Notes. □