Linear Algebra 1

Lecture #10

Linear functions (part I)

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- This lecture has three parts:

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    1. Linear functions: definition, examples, and basic properties

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    2. The image and kernel of a linear function

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    3. The rank-nullity theorem for linear functions

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1. Linear functions: definition, examples, and basic properties

- We have already studied linear functions from $\mathbb{F}^m$ to $\mathbb{F}^n$, for a field $\mathbb{F}$.
- The concept of a linear function can easily be extended to a more general setting, that of arbitrary vector spaces, as follows.

Definition

Given vector spaces U and V over a field $\mathbb{F}$, we say that a function $f : U \to V$ is *linear* provided it satisfies the following two conditions (axioms):

1. $\forall \mathbf{u}_1, \mathbf{u}_2 \in U$: $f(\mathbf{u}_1 + \mathbf{u}_2) = f(\mathbf{u}_1) + f(\mathbf{u}_2)$;
2. $\forall \mathbf{u} \in U$, $\alpha \in \mathbb{F}$: $f(\alpha \mathbf{u}) = \alpha f(\mathbf{u})$.

If the linear function f is also a bijection, then we say that it is an *isomorphism*, and that the vector spaces U and V are *isomorphic*. Linear functions are also called *linear transformations*.

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If the linear function f is also a bijection, then we say that it is an *isomorphism*, and that the vector spaces U and V are *isomorphic*. Linear functions are also called *linear transformations*.

- **Remark:** In the definition of a linear function, the two vector spaces (the domain and the codomain of the function) must be over the same field $\mathbb{F}$.

Example 4.1.1

Let $\mathbb{P}_\mathbb{R}$ be the real vector space of all polynomials with coefficients in $\mathbb{R}$. Show that the function $D : \mathbb{P}_\mathbb{R} \to \mathbb{P}_\mathbb{R}$ given by

$$D\left(\sum_{k=0}^{n} a_k x^k\right) = \sum_{k=1}^{n} k a_k x^{k-1}$$

for all integers $n \geq 0$ and real numbers $a_0, \ldots, a_k$, is linear.

Solution.

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1. Fix $p(x), q(x) \in \mathbb{P}_\mathbb{R}$. Then there exists an integer $n \geq 0$ and real numbers $a_0, \ldots, a_n, b_0, \ldots, b_n$ s.t.

$$p(x) = \sum_{k=0}^{n} a_k x^k \quad \text{and} \quad q(x) = \sum_{k=0}^{n} b_k x^k.$$

Solution (continued). We now compute:

$$
\begin{aligned}
D(p(x)+q(x)) &= D\Big(\big(\sum_{k=0}^{n} a_k x^k\big) + \big(\sum_{k=0}^{n} b_k x^k\big)\Big) \\
&= D\Big(\sum_{k=0}^{n} (a_k + b_k) x^k\Big) \\
&= \sum_{k=1}^{n} k(a_k + b_k) x^{k-1} \\
&= \Big(\sum_{k=1}^{n} k a_k x^{k-1}\Big) + \Big(\sum_{k=1}^{n} k b_k x^{k-1}\Big) \\
&= D\Big(\sum_{k=0}^{n} a_k x^k\Big) + D\Big(\sum_{k=0}^{n} a_k x^k\Big) \\
&= D(p(x)) + D(q(x)).
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□

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Example 4.1.2

Let $\mathrm{Diff}(\mathbb{R})$ be the real vector space of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, and let $\mathrm{Func}(\mathbb{R})$ be the real vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Show that the function $D : \mathrm{Diff}(\mathbb{R}) \to \mathrm{Func}(\mathbb{R})$ given by $D(f) = f'$ for all $f \in \mathrm{Diff}(\mathbb{R})$ is linear. (As usual, f' denotes the derivative of f.)

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Solution. 1. Fix $f, g \in \text{Diff}(\mathbb{R})$. Then by the properties of the derivative, we have that
$D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$.

2. Fix $f \in \text{Diff}(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then by the properties of the derivative, we have that $D(\alpha f) = (\alpha f)' = \alpha f' = \alpha D(f)$.

From 1. and 2., we conclude that D is linear. □

- We now give a few basic properties of linear functions.
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- For the most part (though not exclusively), these are generalizations of the results that we proved previously for linear functions $f : \mathbb{F}^m \to \mathbb{F}^n$ (where $\mathbb{F}$ is a field).
- Most of the results readily generalize to linear functions between arbitrary vectors spaces (over the same field), with one exception: linear functions between general vector spaces do not have standard matrices.
  - It is in fact possible to define the matrix of a linear function between non-trivial, finite-dimensional vectors spaces, but such matrices depend on the particular choice of basis for the domain and codomain.

Theorem 4.1.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a function. Then the following are equivalent:

(i) f is linear;

(ii) for all vectors $\mathbf{u}_1, \mathbf{u}_2 \in U$ and scalars $\alpha_1, \alpha_2 \in \mathbb{F}$, we have that

$$f(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2) \;\; = \;\; \alpha_1 f(\mathbf{u}_1) + \alpha_2 f(\mathbf{u}_2).$$

Proof. Exercise. □

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Proof. Exercise. □

Proposition 4.1.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then for all $\mathbf{u}_1, \mathbf{u}_2 \in U$, we have that

$$f(\mathbf{u}_1 - \mathbf{u}_2) \quad = \quad f(\mathbf{u}_1) - f(\mathbf{u}_2).$$

Proof. Lecture Notes. □

Proposition 4.1.5

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then for all vectors $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$ and all scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$, we have that

$$f\Big(\sum_{i=1}^{k} \alpha_i \mathbf{u}_i\Big) \;=\; \sum_{i=1}^{k} \alpha_i f(\mathbf{u}_i),$$

of, written in another way, that

$$f\Big(\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k\Big) \;=\; \alpha_1 f(\mathbf{u}_1) + \dots + \alpha_k f(\mathbf{u}_k).$$

Proof.

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Proof. This follows from the definition of a linear function via an easy induction on k. The details are left as an exercise. □

Proposition 4.1.6

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then $f(\mathbf{0}) = \mathbf{0}$.

Proof. We observe that

$$f(\mathbf{0}) \overset{(*)}{=} f(0 \cdot \mathbf{0}) \overset{(**)}{=} 0f(\mathbf{0}) \overset{(*)}{=} \mathbf{0},$$

where both instances of (*) follows from Proposition 3.1.3(a), and (**) follows from the fact that f is linear. $\square$

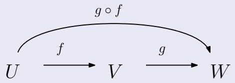
Proposition 4.1.7

Let U, V, and W be vector spaces over a field $\mathbb{F}$. Then all the following hold:

(a) for all linear functions $f, g : U \to V$, the function $f + g$ is linear;[a]

(b) for all linear functions $f : U \to V$ and scalars $\alpha \in \mathbb{F}$, the function $\alpha f : U \to V$ is linear;[b]

(c) for all linear functions $f : U \to V$ and $g : V \to W$, the function $g \circ f$ is liner.[c]



[a] As usual, the function $f + g : U \to V$ is defined by $(f + g)(\mathbf{u}) = f(\mathbf{u}) + g(\mathbf{u})$ for all $\mathbf{u} \in U$.

[b] As usual, the function $\alpha f : U \to V$ is defined by $(\alpha f)(\mathbf{u}) = \alpha\big(f(\mathbf{u})\big)$ for all $\mathbf{u} \in U$.

[c] As usual, the function $g \circ f : U \to W$ is defined by $(g \circ f)(\mathbf{u}) = g\big(f(\mathbf{u})\big)$ for all $\mathbf{u} \in U$.

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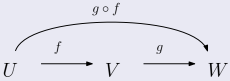

- The proofs of (a) and (b) are left as an exercise.
- The proof of (c) relies on the appropriate definitions and is in the Lecture Notes.

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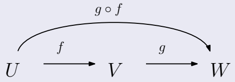

- Given vector spaces U and V over a field $\mathbb{F}$, the set of all linear functions from U to V is denoted by $\mathrm{Hom}(U, V)$.
  - Linear functions are sometimes called “homomorphisms,” which is where the notation $\mathrm{Hom}(U, V)$ comes from.

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- We note that $\mathrm{Hom}(U, V)$ is a vector space over $\mathbb{F}$.
  - The vector addition and scalar multiplication operations in $\mathrm{Hom}(U, V)$ are the addition and scalar multiplication of functions; by parts (a) and (b) of Proposition 4.1.7, $\mathrm{Hom}(U, V)$ is indeed closed under the addition and scalar multiplication of functions.

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  - The zero vector in $\text{Hom}(U, V)$ is the zero function, i.e. the function $f_0 : U \to V$ given by $f_0(\mathbf{u}) = \mathbf{0}_V$ for all $\mathbf{u} \in U$, where $\mathbf{0}_V$ is the zero of the vector space V.

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  - In this case, for $B' \subseteq B$, the notation $f^{-1}[B']$ can be interpreted in two ways: as the preimage of B' under f, and as the image of B' under the inverse function f^{-1}.
  - However, in both cases, $f^{-1}[B']$ is the same subset of A, which is why we usually do not need to specify which interpretation we have in mind.

Definition

Given a **linear** function $f : U \to V$, where U and V are vector spaces over a field $\mathbb{F}$, the *kernel* of f is defined to be the set

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- We further note the kernel is only defined for **linear** functions, and not for general functions.

- Proposition 4.2.1 (next slide) gives the correspondence between the image and kernel of the linear function on the one hand, and the column and null space of the standard matrix on the other hand.
  - Note, however, that the image and kernel are defined for all linear functions, not just those from $\mathbb{F}^m$ to $\mathbb{F}^n$.

Proposition 4.2.1

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, and let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of f. Then both the following hold:

(a) $\text{Im}(f) = \text{Col}(A)$;

(b) $\text{Ker}(f) = \text{Nul}(A)$.

Proof.

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Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, and let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of f. Then both the following hold:

- (a) $\text{Im}(f) = \text{Col}(A)$;
- (b) $\text{Ker}(f) = \text{Nul}(A)$.

Proof. For (a), we observe that

$$\text{Col}(A) \overset{(*)}{=} \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\} \overset{(**)}{=} \{f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{F}^m\} = \text{Im}(f),$$

where (*) follows from Proposition 3.3.2(a), and (**) follows from the fact that A is the standard matrix of f.

Proposition 4.2.1

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, and let $A \in \mathbb{F}^{n\times m}$ be the standard matrix of f. Then both the following hold:

(a) $\text{Im}(f) = \text{Col}(A)$;

(b) $\text{Ker}(f) = \text{Nul}(A)$.

Proof (continued). For (b), we observe that

$$\begin{aligned} \text{Nul}(A) &= \{\mathbf{x} \in \mathbb{F}^m \mid A\mathbf{x} = \mathbf{0}\} \\ &\overset{(*)}{=} \{\mathbf{x} \in \mathbb{F}^m \mid f(\mathbf{x}) = \mathbf{0}\} \\ &= \text{Ker}(f), \end{aligned}$$

where (*) follows from the fact that A is the standard matrix of f. $\square$

Example 4.2.2

Let $\mathbb{P}_{\mathbb{R}}$ be the real vector space of all polynomials with coefficients in $\mathbb{R}$. Consider the function $D : \mathbb{P}_{\mathbb{R}} \to \mathbb{P}_{\mathbb{R}}$ given by

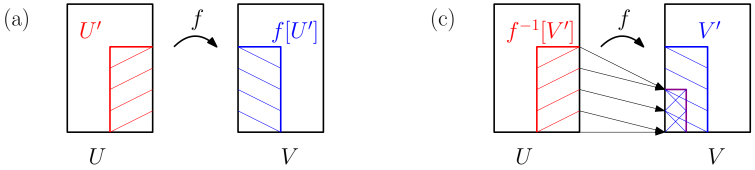
$$D(\sum_{k=0}^{n} a_k x^k) \;=\; \sum_{k=1}^{n} k a_k x^{k-1}$$

for all integers $n \geq 0$ and real numbers $a_0, \ldots, a_k$. By Example 4.1.1, D is linear. Clearly, $\mathrm{Ker}(D)$ is the set of all constant polynomials, and $\mathrm{Im}(D)$ is the set of all polynomials (i.e. $\mathrm{Im}(D) = \mathbb{P}_{\mathbb{R}}$).

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

- (a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;
- (b) $\mathrm{Im}(f)$ is a subspace of V;
- (c) for all subspaces V' of V, we have that $f^{-1}[V']$ is a subspace of U;
- (d) $\mathrm{Ker}(f)$ is a subspace of U.



Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

(b) $\text{Im}(f)$ is a subspace of V;

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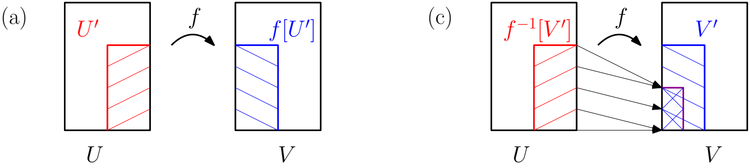
(d) $\text{Ker}(f)$ is a subspace of U.



- The proof relies on Theorem 3.1.7.
- So, let us recall Theorem 3.1.7 (next slide).

Theorem 3.1.7

Let V be a vector space over a field $\mathbb{F}$, and let $U \subseteq V$. Then U is a subspace of V iff the following three conditions are satisfied:

1. $\mathbf{0} \in U$;[a]
2. U is closed under vector addition, that is, for all $\mathbf{u}, \mathbf{v} \in U$, we have that $\mathbf{u} + \mathbf{v} \in U$;
3. U is closed under scalar multiplication, that is, for all $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{u} \in U$.

---

[a] Here, $\mathbf{0}$ is the zero vector in the vector space V.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;
b) $\text{Im}(f)$ is a subspace of V;
c) for all subspaces V' of V, we have that $f^{-1}[V']$ is a subspace of U;
d) $\text{Ker}(f)$ is a subspace of U.

Proof.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

(b) $\text{Im}(f)$ is a subspace of V;

(c) for all subspaces V' of V, we have that $f^{-1}[V']$ is a subspace of U;

(d) $\text{Ker}(f)$ is a subspace of U.

Proof. Since U is a subspace of itself, (a) implies (b).

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

(b) $\text{Im}(f)$ is a subspace of V;

(c) for all subspaces V' of V, we have that $f^{-1}[V']$ is a subspace of U;

(d) $\text{Ker}(f)$ is a subspace of U.

Proof. Since U is a subspace of itself, (a) implies (b). Similarly, since $\text{Ker}(f) = f^{-1}[\{\mathbf{0}\}]$ and $\{\mathbf{0}\}$ is a subspace of V, we have that (c) implies (d).

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

(b) $\mathrm{Im}(f)$ is a subspace of V;

(c) for all subspaces V' of V, we have that $f^{-1}[V']$ is a subspace of U;

(d) $\mathrm{Ker}(f)$ is a subspace of U.

Proof. Since U is a subspace of itself, (a) implies (b). Similarly, since $\mathrm{Ker}(f) = f^{-1}[\{\mathbf{0}\}]$ and $\{\mathbf{0}\}$ is a subspace of V, we have that (c) implies (d). So, it suffices to prove (a) and (c).

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

- (a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;
- (b) $\text{Im}(f)$ is a subspace of V;
- (c) for all subspaces V' of V, we have that $f^{-1}[V']$ is a subspace of U;
- (d) $\text{Ker}(f)$ is a subspace of U.

Proof. Since U is a subspace of itself, (a) implies (b). Similarly, since $\text{Ker}(f) = f^{-1}[\{\mathbf{0}\}]$ and $\{\mathbf{0}\}$ is a subspace of V, we have that (c) implies (d). So, it suffices to prove (a) and (c).

We prove (a). The proof of (c) is similar (see the Lecture Notes).

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a).

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a). Fix a subspace U' of U. WTS $f[U']$ is a subspace of V.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a). Fix a subspace U' of U. WTS $f[U']$ is a subspace of V. Since $f : U \to V$ and $U' \subseteq U$, we have that $f[U'] \subseteq V$.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a). Fix a subspace U' of U. WTS $f[U']$ is a subspace of V. Since $f : U \to V$ and $U' \subseteq U$, we have that $f[U'] \subseteq V$. In view of Theorem 3.1.7, it now suffices to prove the following:

(i) $\mathbf{0}_V \in f[U']$;

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$;

(iii) for all $\mathbf{v} \in f[U']$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{v} \in f[U']$.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a). Fix a subspace U' of U. WTS $f[U']$ is a subspace of V. Since $f : U \to V$ and $U' \subseteq U$, we have that $f[U'] \subseteq V$. In view of Theorem 3.1.7, it now suffices to prove the following:

(i) $\mathbf{0}_V \in f[U']$;

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$;

(iii) for all $\mathbf{v} \in f[U']$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{v} \in f[U']$.

We first prove (i).

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a). Fix a subspace U' of U. WTS $f[U']$ is a subspace of V. Since $f : U \to V$ and $U' \subseteq U$, we have that $f[U'] \subseteq V$. In view of Theorem 3.1.7, it now suffices to prove the following:

(i) $\mathbf{0}_V \in f[U']$;

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$;

(iii) for all $\mathbf{v} \in f[U']$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{v} \in f[U']$.

We first prove (i). Since U' is a subspace of U, Theorem 3.1.7 guarantees that $\mathbf{0}_U \in U'$.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a). Fix a subspace U' of U. WTS $f[U']$ is a subspace of V. Since $f : U \to V$ and $U' \subseteq U$, we have that $f[U'] \subseteq V$. In view of Theorem 3.1.7, it now suffices to prove the following:

(i) $\mathbf{0}_V \in f[U']$;

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$;

(iii) for all $\mathbf{v} \in f[U']$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{v} \in f[U']$.

We first prove (i). Since U' is a subspace of U, Theorem 3.1.7 guarantees that $\mathbf{0}_U \in U'$. On the other hand, by Proposition 4.1.6, we have that $f(\mathbf{0}_U) = \mathbf{0}_V$, and it follows that $\mathbf{0}_V \in f[U']$.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a). Fix a subspace U' of U. WTS $f[U']$ is a subspace of V. Since $f : U \to V$ and $U' \subseteq U$, we have that $f[U'] \subseteq V$. In view of Theorem 3.1.7, it now suffices to prove the following:

(i) $\mathbf{0}_V \in f[U']$;

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$;

(iii) for all $\mathbf{v} \in f[U']$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{v} \in f[U']$.

We first prove (i). Since U' is a subspace of U, Theorem 3.1.7 guarantees that $\mathbf{0}_U \in U'$. On the other hand, by Proposition 4.1.6, we have that $f(\mathbf{0}_U) = \mathbf{0}_V$, and it follows that $\mathbf{0}_V \in f[U']$. This proves (i).

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a) (continued). Next, we prove (ii).

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a) (continued). Next, we prove (ii).

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$

Fix $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$; WTS $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a) (continued). Next, we prove (ii).

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$

Fix $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$; WTS $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$. Since $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we know that $\exists \mathbf{u}_1, \mathbf{u}_2 \in U'$ s.t. $\mathbf{v}_1 = f(\mathbf{u}_1)$ and $\mathbf{v}_2 = f(\mathbf{u}_2)$.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a) (continued). Next, we prove (ii).

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$

Fix $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$; WTS $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$. Since $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we know that $\exists \mathbf{u}_1, \mathbf{u}_2 \in U'$ s.t. $\mathbf{v}_1 = f(\mathbf{u}_1)$ and $\mathbf{v}_2 = f(\mathbf{u}_2)$. Since U' is a subspace of U, we have that $\mathbf{u}_1 + \mathbf{u}_2 \in U'$.

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

(a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a) (continued). Next, we prove (ii).

(ii) for all $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we have that $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$

Fix $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$; WTS $\mathbf{v}_1 + \mathbf{v}_2 \in f[U']$. Since $\mathbf{v}_1, \mathbf{v}_2 \in f[U']$, we know that $\exists \mathbf{u}_1, \mathbf{u}_2 \in U'$ s.t. $\mathbf{v}_1 = f(\mathbf{u}_1)$ and $\mathbf{v}_2 = f(\mathbf{u}_2)$. Since U' is a subspace of U, we have that $\mathbf{u}_1 + \mathbf{u}_2 \in U'$. But now we have that

$$\mathbf{v}_1 + \mathbf{v}_2 \quad = \quad f(\mathbf{u}_1) + f(\mathbf{u}_2) \quad \overset{(*)}{=} \quad f(\mathbf{u}_1 + \mathbf{u}_2) \quad \overset{(**)}{\in} \quad f[U'],$$

where (*) follows from the fact that f is linear, and (**) follows from the fact that $\mathbf{u}_1 + \mathbf{u}_2 \in U'$. This proves (ii).

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

- (a) for all subspaces U' of U, we have that $f[U']$ is a subspace of V;

Proof of (a) (continued).

- (iii) for all $\mathbf{v} \in f[U']$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{v} \in f[U']$.

The proof of (iii) is similar to the proof of (ii) (details: Lecture Notes). This proves (a). □

Theorem 4.2.3

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then all the following hold:

a. for all subspaces U' of U, we have that $f[U']$ is a subspace of V;
b. $\text{Im}(f)$ is a subspace of V;
c. for all subspaces V' of V, we have that $f^{-1}[V']$ is a subspace of U;
d. $\text{Ker}(f)$ is a subspace of U.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

Proof.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

Proof. To avoid any possible confusion, we denote by $\mathbf{0}_U$ the zero vector of the vector space U, and we denote by $\mathbf{0}_V$ the zero vector of the vector space V. We need to show that f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}_U\}$.

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Suppose first that f is one-to-one.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

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Suppose first that f is one-to-one. By Proposition 4.1.6, we have that $f(\mathbf{0}_U) = \mathbf{0}_V$,

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\mathrm{Ker}(f) = \{\mathbf{0}\}$.

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Suppose first that f is one-to-one. By Proposition 4.1.6, we have that $f(\mathbf{0}_U) = \mathbf{0}_V$, and it follows that $\mathbf{0}_U \in \mathrm{Ker}(f)$.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

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Suppose first that f is one-to-one. By Proposition 4.1.6, we have that $f(\mathbf{0}_U) = \mathbf{0}_V$, and it follows that $\mathbf{0}_U \in \text{Ker}(f)$. It remains to show that $\mathbf{0}_U$ is the **only** element of $\text{Ker}(f)$.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

Proof. To avoid any possible confusion, we denote by $\mathbf{0}_U$ the zero vector of the vector space U, and we denote by $\mathbf{0}_V$ the zero vector of the vector space V. We need to show that f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}_U\}$.

Suppose first that f is one-to-one. By Proposition 4.1.6, we have that $f(\mathbf{0}_U) = \mathbf{0}_V$, and it follows that $\mathbf{0}_U \in \text{Ker}(f)$. It remains to show that $\mathbf{0}_U$ is the **only** element of $\text{Ker}(f)$. So, fix any $\mathbf{u} \in \text{Ker}(f)$.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

Proof. To avoid any possible confusion, we denote by $\mathbf{0}_U$ the zero vector of the vector space U, and we denote by $\mathbf{0}_V$ the zero vector of the vector space V. We need to show that f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}_U\}$.

Suppose first that f is one-to-one. By Proposition 4.1.6, we have that $f(\mathbf{0}_U) = \mathbf{0}_V$, and it follows that $\mathbf{0}_U \in \text{Ker}(f)$. It remains to show that $\mathbf{0}_U$ is the **only** element of $\text{Ker}(f)$. So, fix any $\mathbf{u} \in \text{Ker}(f)$. Then $f(\mathbf{u}) = \mathbf{0}_V = f(\mathbf{0}_U)$,

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

Proof. To avoid any possible confusion, we denote by $\mathbf{0}_U$ the zero vector of the vector space U, and we denote by $\mathbf{0}_V$ the zero vector of the vector space V. We need to show that f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}_U\}$.

Suppose first that f is one-to-one. By Proposition 4.1.6, we have that $f(\mathbf{0}_U) = \mathbf{0}_V$, and it follows that $\mathbf{0}_U \in \text{Ker}(f)$. It remains to show that $\mathbf{0}_U$ is the **only** element of $\text{Ker}(f)$. So, fix any $\mathbf{u} \in \text{Ker}(f)$. Then $f(\mathbf{u}) = \mathbf{0}_V = f(\mathbf{0}_U)$, and so since f is one-to-one, we have that $\mathbf{u} = \mathbf{0}_U$.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

Proof. To avoid any possible confusion, we denote by $\mathbf{0}_U$ the zero vector of the vector space U, and we denote by $\mathbf{0}_V$ the zero vector of the vector space V. We need to show that f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}_U\}$.

Suppose first that f is one-to-one. By Proposition 4.1.6, we have that $f(\mathbf{0}_U) = \mathbf{0}_V$, and it follows that $\mathbf{0}_U \in \text{Ker}(f)$. It remains to show that $\mathbf{0}_U$ is the **only** element of $\text{Ker}(f)$. So, fix any $\mathbf{u} \in \text{Ker}(f)$. Then $f(\mathbf{u}) = \mathbf{0}_V = f(\mathbf{0}_U)$, and so since f is one-to-one, we have that $\mathbf{u} = \mathbf{0}_U$. This proves that $\text{Ker}(f) = \{\mathbf{0}_U\}$.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\mathrm{Ker}(f) = \{\mathbf{0}\}$.

Proof (continued). Suppose now that $\mathrm{Ker}(f) = \{\mathbf{0}_U\}$.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

Proof (continued). Suppose now that $\text{Ker}(f) = \{\mathbf{0}_U\}$. Fix $\mathbf{u}_1, \mathbf{u}_2 \in U$, and assume that $f(\mathbf{u}_1) = f(\mathbf{u}_2)$; WTS $\mathbf{u}_1 = \mathbf{u}_2$.

Theorem 4.2.4

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.

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$$f(\mathbf{u}_1 - \mathbf{u}_2) \stackrel{(*)}{=} f(\mathbf{u}_1) - f(\mathbf{u}_2) \stackrel{(**)}{=} \mathbf{0}_V,$$

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3. The rank-nullity theorem for linear functions

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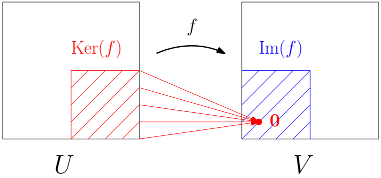

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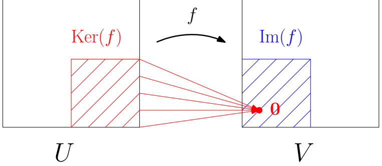

- The *rank* of f is defined to be

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and the *nullity* of f is $\dim(\mathrm{Ker}(f))$.

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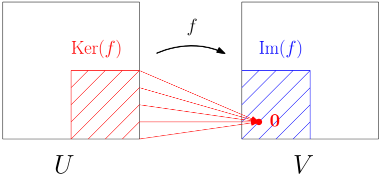
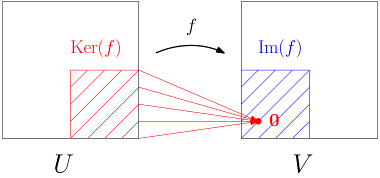


- The *rank* of f is defined to be

$$\mathrm{rank}(f) := \dim(\mathrm{Im}(f)),$$

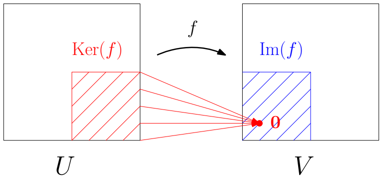
and the *nullity* of f is $\dim(\mathrm{Ker}(f))$.
- We note that both the rank and the nullity of f may possibly be infinite.



Proposition 4.2.5

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then $\text{rank}(f) \le \dim(V)$.

Proof.



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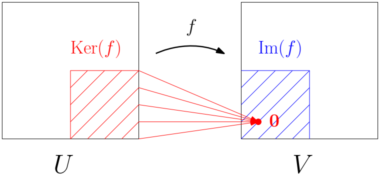

Proposition 4.2.5

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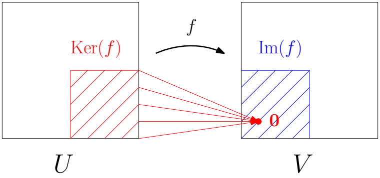
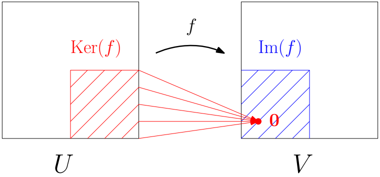
Proof. We may assume that $n := \dim(V)$ is finite, for otherwise, this is immediate. By Theorem 4.2.3, $\text{Im}(f)$ is a subspace of V,



Proposition 4.2.5

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then $\text{rank}(f) \leq \dim(V)$.

Proof. We may assume that $n := \dim(V)$ is finite, for otherwise, this is immediate. By Theorem 4.2.3, $\text{Im}(f)$ is a subspace of V, and so by Theorem 3.2.21, we have that $\dim\big(\text{Im}(f)\big) \leq \dim(V)$, i.e. $\text{rank}(f) \leq \dim(V)$. $\square$



- Reminder:

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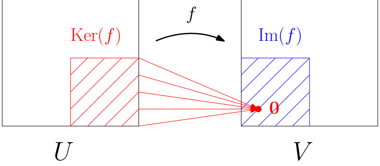
Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then f is one-to-one iff $\text{Ker}(f) = \{\mathbf{0}\}$.



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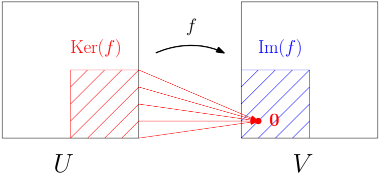
- For onto-ness, we have the following theorem (when the codomain is finite-dimensional):



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- For onto-ness, we have the following theorem (when the codomain is finite-dimensional):

Proposition 4.2.6

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Assume that V is **finite-dimensional**. Then f is onto iff $\text{rank}(f) = \dim(V)$.

Proposition 4.2.6

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Assume that V is **finite-dimensional**. Then f is onto iff $\text{rank}(f) = \dim(V)$.

Proof.

Proposition 4.2.6

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Assume that V is **finite-dimensional**. Then f is onto iff $\text{rank}(f) = \dim(V)$.

Proof. We have the following sequence of equivalent statements:

$$\begin{aligned} f \text{ is onto} \quad &\overset{(*)}{\iff} \quad \text{Im}(f) = V \\ &\overset{(**)}{\iff} \quad \dim\big(\text{Im}(f)\big) = \dim(V) \\ &\overset{(***)}{\iff} \quad \text{rank}(f) = \dim(V), \end{aligned}$$

where (*) follows from the definition of an onto function, (**) follows from Theorem 3.2.21 (since $\text{Im}(f)$ is a subspace of V, and V is finite-dimensional), and (***) follows from the definition of rank. □

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- **Warning:** Proposition 4.2.6 only applies to linear functions that have a **finite-dimensional codomain**.
  - Do not apply Proposition 4.2.6 to linear functions with an infinite-dimensional codomain!

- Reminder:

Proposition 4.2.1

Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, and let $A \in \mathbb{F}^{n \times m}$ be the standard matrix of f. Then both the following hold:

(a) $\text{Im}(f) = \text{Col}(A)$;

(b) $\text{Ker}(f) = \text{Nul}(A)$.

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a. $\text{rank}(f) = \text{rank}(A)$;
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Proof.

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b. $\dim\big(\text{Ker}(f)\big) = \dim\big(\text{Nul}(A)\big)$.

Proof. By Proposition 4.2.1, we have that $\text{Im}(f) = \text{Col}(A)$ and $\text{Ker}(f) = \text{Nul}(A)$.

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Proof. By Proposition 4.2.1, we have that $\text{Im}(f) = \text{Col}(A)$ and $\text{Ker}(f) = \text{Nul}(A)$. The latter immediately implies (b).

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Let $\mathbb{F}$ be a field, let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function, and let $A \in \mathbb{F}^{n\times m}$ be the standard matrix of f. Then both the following hold:

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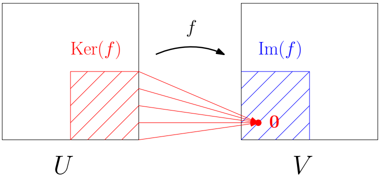
Proof. By Proposition 4.2.1, we have that $\text{Im}(f) = \text{Col}(A)$ and $\text{Ker}(f) = \text{Nul}(A)$. The latter immediately implies (b). For (a), we observe that

$$\text{rank}(f) \;=\; \dim\big(\text{Im}(f)\big) \;=\; \dim\big(\text{Col}(A)\big) \;\overset{(*)}{=}\; \text{rank}(A),$$

where (*) follows from Theorem 3.3.4. □

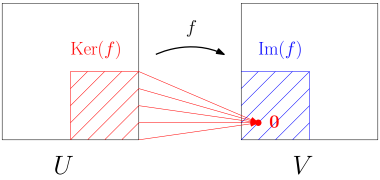


The rank–nullity theorem (linear function version)

Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Then every linear function $f : U \to V$ satisfies

$$\text{rank}(f) + \dim\big(\text{Ker}(f)\big) \quad = \quad \dim(U),$$

and in particular, both $\text{Ker}(f)$ and $\text{Im}(f)$ are finite-dimensional.



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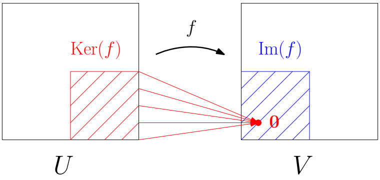
- Proof: Later!



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and in particular, both $\operatorname{Ker}(f)$ and $\operatorname{Im}(f)$ are finite-dimensional.

- Proof: Later!
- First, we show how the rank-nullity theorem for linear functions implies the rank-nullity theorem for matrices.

The rank–nullity theorem (matrix version)

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then

$$\operatorname{rank}(A) + \dim(\operatorname{Nul}(A)) \quad = \quad \underbrace{m}_{\substack{= \text{ number of} \\ \text{columns of } A}} \quad .$$

Proof (using the rank-nullity theorem for linear functions).

The rank–nullity theorem (matrix version)

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then

$$\operatorname{rank}(A) + \dim(\operatorname{Nul}(A)) = \underbrace{m}_{\substack{= \text{ number of} \\ \text{columns of } A}} .$$

Proof (using the rank-nullity theorem for linear functions). Let $f : \mathbb{F}^m \to \mathbb{F}^n$ be given by $f(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{F}^m$.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then

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Proof (using the rank-nullity theorem for linear functions). Let $f : \mathbb{F}^m \to \mathbb{F}^n$ be given by $f(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{F}^m$. By Proposition 1.10.4, f is linear, and obviously, A is the standard matrix of f.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then

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Proof (using the rank-nullity theorem for linear functions). Let $f : \mathbb{F}^m \to \mathbb{F}^n$ be given by $f(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{F}^m$. By Proposition 1.10.4, f is linear, and obviously, A is the standard matrix of f. We now have that

$$\begin{aligned} \text{rank}(A) + \dim\big(\text{Nul}(A)\big) \quad &\overset{(*)}{=} \quad \text{rank}(f) + \dim\big(\text{Ker}(f)\big) \\ &\overset{(**)}{=} \quad \dim(\mathbb{F}^m) \quad = \quad m, \end{aligned}$$

where (*) follows from Proposition 4.2.7, and (**) follows from the rank-nullity theorem for linear functions. □

The rank–nullity theorem (linear function version)

Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Then every linear function $f : U \to V$ satisfies

$$\operatorname{rank}(f) + \dim\big(\operatorname{Ker}(f)\big) \quad = \quad \dim(U),$$

and in particular, both $\operatorname{Ker}(f)$ and $\operatorname{Im}(f)$ are finite-dimensional.

Proof.

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Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Then every linear function $f : U \to V$ satisfies

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and in particular, both $\operatorname{Ker}(f)$ and $\operatorname{Im}(f)$ are finite-dimensional.

Proof. By Theorem 4.2.3, $\operatorname{Ker}(f)$ is a subspace of U, and $\operatorname{Im}(f)$ is a subspace of V.

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Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Then every linear function $f : U \to V$ satisfies

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and in particular, both $\text{Ker}(f)$ and $\text{Im}(f)$ are finite-dimensional.

Proof. By Theorem 4.2.3, $\text{Ker}(f)$ is a subspace of U, and $\text{Im}(f)$ is a subspace of V. Next, since U is finite-dimensional, Theorem 3.2.21 guarantees that its subspace $\text{Ker}(f)$ is finite-dimensional and satisfies $\dim\big(\text{Ker}(f)\big) \leq \dim(U)$.

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Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Then every linear function $f : U \to V$ satisfies

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Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Then every linear function $f : U \to V$ satisfies

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Proof. By Theorem 4.2.3, $\text{Ker}(f)$ is a subspace of U, and $\text{Im}(f)$ is a subspace of V. Next, since U is finite-dimensional, Theorem 3.2.21 guarantees that its subspace $\text{Ker}(f)$ is finite-dimensional and satisfies $\dim\big(\text{Ker}(f)\big) \leq \dim(U)$. Set $k := \dim\big(\text{Ker}(f)\big)$ and $m := \dim(U)$ (so, $k \leq m$). By definition, we have that $\text{rank}(f) = \dim\big(\text{Im}(f)\big)$. Thus, to complete the proof, we need only exhibit a basis of $\text{Im}(f)$ of size $m - k$.

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$$\text{rank}(f) + \dim\big(\text{Ker}(f)\big) \quad = \quad \dim(U),$$

and in particular, both $\text{Ker}(f)$ and $\text{Im}(f)$ are finite-dimensional.

Proof. By Theorem 4.2.3, $\text{Ker}(f)$ is a subspace of U, and $\text{Im}(f)$ is a subspace of V. Next, since U is finite-dimensional, Theorem 3.2.21 guarantees that its subspace $\text{Ker}(f)$ is finite-dimensional and satisfies $\dim\big(\text{Ker}(f)\big) \leq \dim(U)$. Set $k := \dim\big(\text{Ker}(f)\big)$ and $m := \dim(U)$ (so, $k \leq m$). By definition, we have that $\text{rank}(f) = \dim\big(\text{Im}(f)\big)$. Thus, to complete the proof, we need only exhibit a basis of $\text{Im}(f)$ of size $m - k$. Indeed, this will imply $\text{rank}(f) = \dim\big(\text{Im}(f)\big) = m - k$, and the result will follow immediately.

Proof (continued). Reminder: $k = \dim(\mathrm{Ker}(f))$, $m = \dim(U)$, $k \leq m$. WTS $\mathrm{Im}(f)$ has a basis of size $m - k$.

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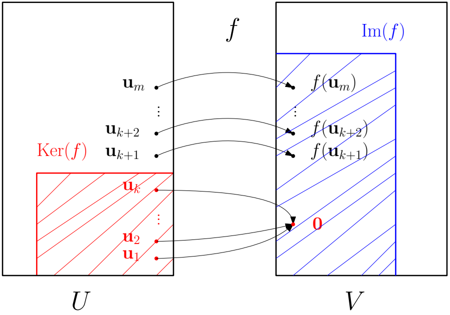
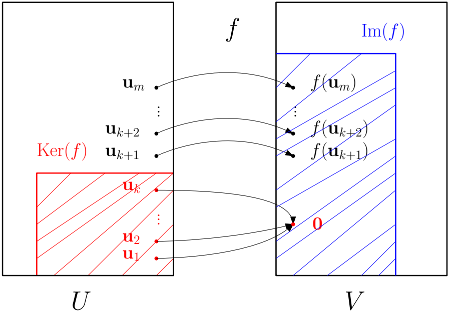


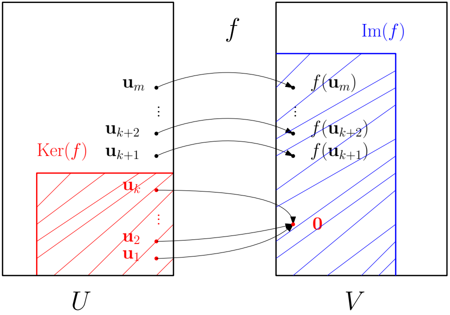
We proceed as follows.

Proof (continued). Reminder: $k = \dim(\mathrm{Ker}(f))$, $m = \dim(U)$, $k \leq m$. WTS $\mathrm{Im}(f)$ has a basis of size $m - k$.



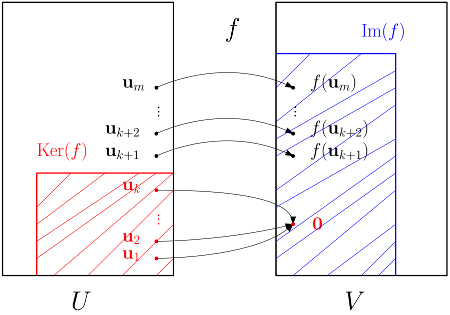
We proceed as follows. Fix a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of $\mathrm{Ker}(f)$.

Proof (continued). Reminder: $k = \dim(\text{Ker}(f))$, $m = \dim(U)$, $k \leq m$. WTS $\text{Im}(f)$ has a basis of size $m - k$.



We proceed as follows. Fix a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of $\text{Ker}(f)$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set in a finite-dimensional vector space U; so, by Theorem 3.2.19, $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ can be extended to a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_m\}$ of U.

Proof (continued). Reminder: $k = \dim(\text{Ker}(f))$, $m = \dim(U)$, $k \leq m$. WTS $\text{Im}(f)$ has a basis of size $m - k$.



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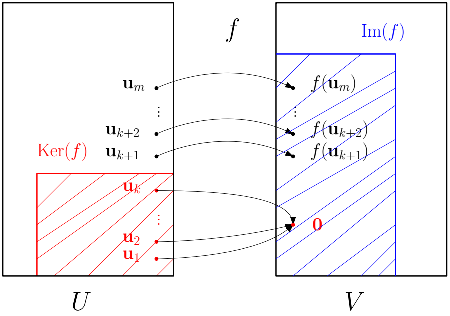
Proof (continued).



It suffices to prove the following two claims.

> ***Claim 1.*** *Vectors $f(\mathbf{u}_{k+1}), \ldots, f(\mathbf{u}_m)$ are linearly independent.*
>
> ***Claim 2.*** *$Im(f) = Span(f(\mathbf{u}_{k+1}), \ldots, f(\mathbf{u}_m))$.*

We prove them one by one.

Claim 1. *Vectors* $f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m)$ *are linearly independent.*

Proof of Claim 1.

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Proof of Claim 1. Fix scalars $\alpha_{k+1}, \dots, \alpha_m \in \mathbb{F}$ s.t.

$$\alpha_{k+1} f(\mathbf{u}_{k+1}) + \dots + \alpha_m f(\mathbf{u}_m) \quad = \quad \mathbf{0}.$$

WTS $\alpha_{k+1} = \dots = \alpha_m = 0$.

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WTS $\alpha_{k+1} = \dots = \alpha_m = 0$. Note that

$$f(\alpha_{k+1}\mathbf{u}_{k+1} + \dots + \alpha_m \mathbf{u}_m) \quad \overset{(*)}{=} \quad \alpha_{k+1} f(\mathbf{u}_{k+1}) + \dots + \alpha_m f(\mathbf{u}_m) \quad = \quad \mathbf{0},$$

where (*) follows from the fact that f is linear (and more precisely, from Proposition 4.1.5).

Claim 1. *Vectors* $f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m)$ *are linearly independent.*

Proof of Claim 1. Fix scalars $\alpha_{k+1}, \dots, \alpha_m \in \mathbb{F}$ s.t.

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where (*) follows from the fact that f is linear (and more precisely, from Proposition 4.1.5). But now we have that $\alpha_{k+1}\mathbf{u}_{k+1} + \dots + \alpha_m \mathbf{u}_m \in \text{Ker}(f)$.

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Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of $\text{Ker}(f)$, we have that $\alpha_{k+1}\mathbf{u}_{k+1} + \dots + \alpha_m \mathbf{u}_m$ is a linear combination of the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$, i.e. $\exists \alpha_1, \dots, \alpha_k \in \mathbb{F}$ s.t.

$$\alpha_{k+1}\mathbf{u}_{k+1} + \dots + \alpha_m \mathbf{u}_m \quad = \quad \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k.$$

Claim 1. *Vectors $f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m)$ are linearly independent.*

Proof of Claim 1 (continued). Reminder:
$\alpha_{k+1}\mathbf{u}_{k+1} + \dots + \alpha_m\mathbf{u}_m = \alpha_1\mathbf{u}_1 + \dots + \alpha_k\mathbf{u}_k$.
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But this implies that

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Since vectors $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_m$ are linearly independent (because they form a basis of U), we deduce that
$-\alpha_1 = \cdots = -\alpha_k = \alpha_{k+1} = \cdots = \alpha_m = 0$.

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In particular, $\alpha_{k+1} = \dots = \alpha_m = 0$, and it follows that vectors $f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m)$ are indeed linearly independent, which is what we needed to show. ♦

Claim 2. *Im*$(f) = $ *Span*$\big(f(\mathbf{u}_{k+1}), \ldots, f(\mathbf{u}_m)\big)$.

Proof of Claim 2.

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Proof of Claim 2. By definition, $f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m) \in \text{Im}(f)$.

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Proof of Claim 2. By definition, $f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m) \in \text{Im}(f)$. Since $\text{Im}(f)$ is a subspace of V (and therefore a vector space in its own right), Theorem 3.1.11 guarantees that $\text{Span}(f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m))$ is a subspace (and in particular, a subset) of $\text{Im}(f)$.

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Proof of Claim 2. By definition, $f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m) \in \text{Im}(f)$. Since $\text{Im}(f)$ is a subspace of V (and therefore a vector space in its own right), Theorem 3.1.11 guarantees that $\text{Span}(f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m))$ is a subspace (and in particular, a subset) of $\text{Im}(f)$. It remains to show that $\text{Im}(f) \subseteq \text{Span}(f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m))$, i.e. that every vector in $\text{Im}(f)$ is a linear combination of the vectors $f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m)$.

Claim 2. $Im(f) = Span\big(f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m)\big)$.

Proof of Claim 2. (continued). Fix $\mathbf{v} \in \text{Im}(f)$.

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Proof of Claim 2. (continued). Fix $\mathbf{v} \in \text{Im}(f)$. Then $\exists \mathbf{u} \in U$ s.t. $\mathbf{v} = f(\mathbf{u})$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a basis of U, we know that $\exists \alpha_1, \dots, \alpha_m \in \mathbb{F}$ s.t. $\mathbf{u} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_m \mathbf{u}_m$.

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$$\begin{aligned} \mathbf{v} = f(\mathbf{u}) &= f(\alpha_1 \mathbf{u}_1 + \dots + \alpha_m \mathbf{u}_m) \\ &\overset{(*)}{=} \alpha_1 f(\mathbf{u}_1) + \dots + \alpha_m f(\mathbf{u}_m) \\ &\overset{(**)}{=} \alpha_{k+1} f(\mathbf{u}_{k+1}) + \dots + \alpha_m f(\mathbf{u}_m), \end{aligned}$$

where (*) follows from the fact that f is linear (and more precisely, from Proposition 4.1.5), and (**) follows from the fact that $f(\mathbf{u}_1) = \dots = f(\mathbf{u}_k) = \mathbf{0}$ (because $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathrm{Ker}(f)$).

Claim 2. *Im*$(f) = Span(f(\mathbf{u}_{k+1}), \dots, f(\mathbf{u}_m))$.

Proof of Claim 2. (continued). Fix $\mathbf{v} \in \text{Im}(f)$. Then $\exists \mathbf{u} \in U$ s.t. $\mathbf{v} = f(\mathbf{u})$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a basis of U, we know that $\exists \alpha_1, \dots, \alpha_m \in \mathbb{F}$ s.t. $\mathbf{u} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_m \mathbf{u}_m$. We now have the following:

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The rank–nullity theorem (linear function version)

Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is finite-dimensional. Then every linear function $f : U \to V$ satisfies

$$\mathrm{rank}(f) + \dim\bigl(\mathrm{Ker}(f)\bigr) \quad = \quad \dim(U),$$

and in particular, both $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are finite-dimensional.

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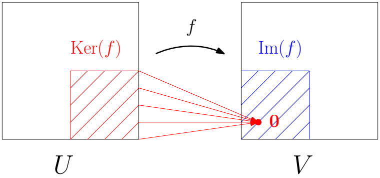
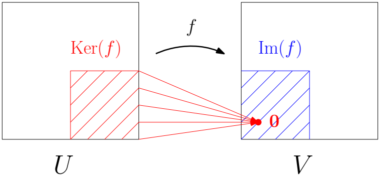
- The rank-nullity theorem for linear functions has a few easy dimension-related corollaries, which we now turn to.



Corollary 4.2.8

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then

$$\text{rank}(f) \quad \leq \quad \min\{\dim(U), \dim(V)\}.$$



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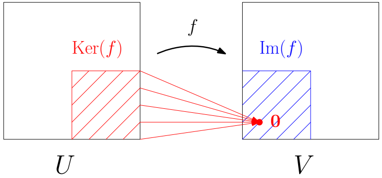
- **Remark:** By definition, $\text{rank}(f) = \dim\big(\text{Im}(f)\big)$.



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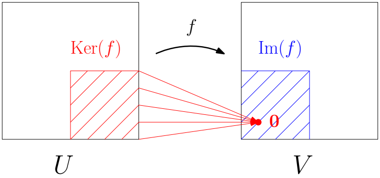
- **Remark:** By definition, $\operatorname{rank}(f) = \dim\big(\operatorname{Im}(f)\big)$.
  - So, Corollary 4.2.8 states that the dimension of the image of a linear function is at most the dimension of the domain and also at most the dimension of the codomain.



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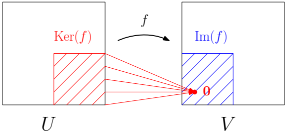
- **Remark:** By definition, $\text{rank}(f) = \dim(\text{Im}(f))$.
  - So, Corollary 4.2.8 states that the dimension of the image of a linear function is at most the dimension of the domain and also at most the dimension of the codomain.
  - We note that in Corollary 4.2.8, vector spaces U and V may possibly be infinite-dimensional.



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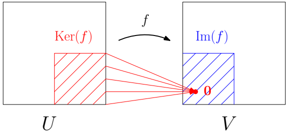
Proof.



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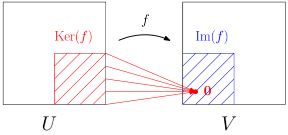
Proof. The fact that $\text{rank}(f) \leq \dim(V)$ follows from Proposition 4.2.5.



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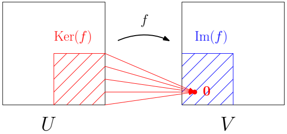
Proof. The fact that $\operatorname{rank}(f) \leq \dim(V)$ follows from Proposition 4.2.5. It remains to show that $\operatorname{rank}(f) \leq \dim(U)$.



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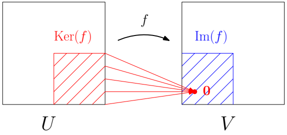
Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then

$$\text{rank}(f) \quad \leq \quad \min\{\dim(U), \dim(V)\}.$$

Proof. The fact that $\text{rank}(f) \leq \dim(V)$ follows from Proposition 4.2.5. It remains to show that $\text{rank}(f) \leq \dim(U)$. If $\dim(U) = \infty$, then this is immediate.



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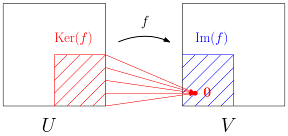

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Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then

$$\text{rank}(f) \quad \leq \quad \min\{\dim(U), \dim(V)\}.$$

Proof. The fact that $\text{rank}(f) \leq \dim(V)$ follows from Proposition 4.2.5. It remains to show that $\text{rank}(f) \leq \dim(U)$. If $\dim(U) = \infty$, then this is immediate. So, let us assume that U is finite-dimensional. Then

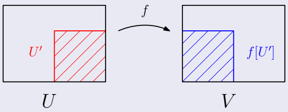
$$\text{rank}(f) \quad \leq \quad \text{rank}(f) + \dim(\text{Ker}(f)) \quad \stackrel{(*)}{=} \quad \dim(U),$$

where (*) follows from the rank-nullity theorem for linear functions. $\square$

Corollary 4.2.9

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then for any subspace U' of U, we have that

$$\dim(f[U']) \quad \leq \quad \min\{\dim(U'), \dim(V)\}.$$

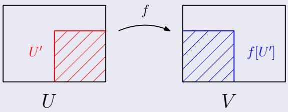


Proof.

Corollary 4.2.9

Let U and V be vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Then for any subspace U' of U, we have that

$$\dim(f[U']) \quad \leq \quad \min\{\dim(U'), \dim(V)\}.$$



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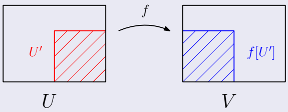

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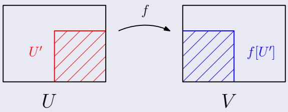

Proof. Consider the function $f' := f \restriction U'$ (the restriction of f to U'). Since U' is a subspace of U and $f : U \to V$ is linear, we have that $f' : U' \to V$ is also linear. So,

$$\begin{aligned}\dim\big(f[U']\big) \quad = \quad \dim\big(f'[U']\big) \quad &= \quad \dim\big(\mathrm{Im}(f')\big) \\ &\overset{(*)}{\leq} \quad \min\big\{\dim(U'), \dim(V)\big\},\end{aligned}$$

where (*) follows from Corollary 4.2.8. □

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  - Similar remarks apply to higher-dimensional generalizations of subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$.

- **Linear functions between vector spaces of the same finite dimension:**

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  - By the Invertible Matrix Theorem, for a linear function $f : \mathbb{F}^n \to \mathbb{F}^n$ (where $\mathbb{F}$ is a field), the following are equivalent:
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  - Here, we assumed that the domain and the codomain of f are the same (namely, $\mathbb{F}^n$).
  - Using Theorem 4.2.4 (which states that a linear function is one-to-one iff its kernel is $\{\mathbf{0}\}$) and the rank-nullity theorem for linear functions, we can generalize this to linear functions between two vector spaces of the same finite dimension (next slide).

Corollary 4.2.10

Let U and V be **finite-dimensional** vector spaces over a field $\mathbb{F}$, and assume that $\dim(U) = \dim(V)$. Let $f : U \to V$ be a linear function. Then the following are equivalent:

(i) f is one-to-one;

(ii) f is onto;

(iii) f is a bijection (and therefore an isomorphism).

- **Warning:** Corollary 4.2.10 only works if U and V (the domain and codomain of our linear function f) are of the same **finite** dimension. Do not attempt to apply the corollary to linear functions between infinite-dimensional vector spaces, or between vector spaces of different dimension.

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Proof.

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We now have the following sequence of equivalent statements (next slide):

Proof (continued).

f is one-to-one	$\iff$	$\text{Ker}(f) = \{\mathbf{0}\}$	by Theorem 4.2.4
	$\iff$	$\dim\big(\text{Ker}(f)\big) = 0$	
	$\iff$	$\text{rank}(f) = \dim(U)$	by the rank-nullity theorem
	$\iff$	$\dim\big(\text{Im}(f)\big) = \dim(U)$	by the definition of $\text{rank}(f)$
	$\iff$	$\dim\big(\text{Im}(f)\big) = \dim(V)$	because $\dim(U) = \dim(V)$
	$\iff$	$\text{Im}(f) = V$	by Theorem 3.2.21, since V is fin.-dim.
	$\iff$	f is onto V.	

So, (i) and (ii) are equivalent. This completes the argument. □

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