Linear Algebra 1

Lecture #8

Vector spaces (part I)

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Definition

Let $\mathbb{F}$ be a field with additive identity 0 and multiplicative identity 1. In what follows, we shall refer to elements of $\mathbb{F}$ as *scalars*. A *vector space* (or *linear space*) over the field $\mathbb{F}$ is a set V, together with a binary operation $+$ on V (called *vector addition*) and an operation $\cdot : \mathbb{F} \times V \to V$ (called *scalar multiplication*), satisfying the following axioms:

1. $(V, +)$ is an abelian group; the identity element of $(V, +)$ is denoted by $\mathbf{0}$ ("zero vector"), and for any vector $\mathbf{v} \in V$, the inverse of $\mathbf{v}$ in $(V, +)$ is dented by $-\mathbf{v}$;
2. for all vectors $\mathbf{v} \in V$, we have $1\mathbf{v} = \mathbf{v}$;
3. for all vectors $\mathbf{v} \in V$ and scalars $\alpha, \beta \in \mathbb{F}$, we have $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$;
4. for all vectors $\mathbf{v} \in V$ and scalars $\alpha, \beta \in \mathbb{F}$, we have $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$;
5. for all vectors $\mathbf{u}, \mathbf{v} \in V$ and scalars $\alpha \in \mathbb{F}$, we have $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

Example 3.1.1

Let $\mathbb{F}$ be a field. Then all the following are vector spaces over $\mathbb{F}$ (in each case, vector addition and scalar multiplication are defined in the natural way):

1. $\mathbb{F}^n$;
2. $\mathbb{F}^{n \times m}$;
3. the set of all functions from $\mathbb{F}$ to $\mathbb{F}$;
4. the set $\mathbb{P}_{\mathbb{F}}$ of all polynomials (in one varialble, typically x) with coefficients in the field $\mathbb{F}$;[a]
    - **Notation:** Some texts use the notation $\mathbb{F}[x]$ instead of $\mathbb{P}_{\mathbb{F}}$ (if x is the variable used in the polynomials in question).
5. for a non-negative integer n, the set $\mathbb{P}_{\mathbb{F}}^n$ of all polynomials of degree at most n and with coefficients in $\mathbb{F}$.[b]

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[a] One could also consider polynomials in more than one variable (say, $x_1, \dots, x_k$) and with coefficients in $\mathbb{F}$. This, too, is a vector space over $\mathbb{F}$.

[b] The notation $\mathbb{P}_{\mathbb{F}}^n$ is not fully standard (there is no fully standard notation for this), but it is the notation that we will use.

- If you have studied calculus, here is another example of a vector space.

Example 3.1.2

The following are real vector spaces (with vector addition and scalar multiplication defined in the usual way):

1. the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$;
2. the set of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

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- **Terminology:**
  - Elements of any vector space are considered vectors (even if they do not "look like" vectors, i.e. even if they are matrices, functions, or polynomials).

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- **Terminology:**
  - Elements of any vector space are considered vectors (even if they do not "look like" vectors, i.e. even if they are matrices, functions, or polynomials).
  - A *real vector space* is a vector space over the field $\mathbb{R}$, and a *complex vector space* is a vector space over the field $\mathbb{C}$.

- For any field $\mathbb{F}$, we have the *trivial* vector space $\{\mathbf{0}\}$ over the field $\mathbb{F}$.
  - In this vector space, vector addition and scalar multiplication are defined in the obvious way: $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $\alpha\mathbf{0} = \mathbf{0}$ for all scalars $\alpha \in \mathbb{F}$.

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- A vector space is *non-trivial* if it contains at least one non-zero vector.

Proposition 3.1.3

Let V be a vector space over a field $\mathbb{F}$. Then all the following hold:

(a) for all $\mathbf{v} \in V$, $0\mathbf{v} = \mathbf{0}$;[a]

(b) for all $\alpha \in \mathbb{F}$, $\alpha\mathbf{0} = \mathbf{0}$;

(c) for all $\mathbf{v} \in V$ and $\alpha \in \mathbb{F}$, if $\alpha\mathbf{v} = \mathbf{0}$, then $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$;

(d) for all $\mathbf{v} \in V$, $(-1)\mathbf{v} = -\mathbf{v}$.[b]

[a] Here, 0 is the zero of the field $\mathbb{F}$, and $\mathbf{0}$ is the zero vector in V.

[b] Here, -1 is the additive inverse of the multiplicative identity of the field $\mathbb{F}$, and in particular, -1 is a scalar. So, $(-1)\mathbf{v}$ is the product of the scalar -1 and the vector $\mathbf{v}$. On the other hand, $-\mathbf{v}$ is the additive inverse of the vector $\mathbf{v}$.

- The proof of (a) is in the Lecture Notes.
- The rest is left as an exercise.

Definition

Let V be a vector space over a field $\mathbb{F}$. A *vector subspace* (or *linear subspace* or simply *subspace*) of V is a set $U \subseteq V$ s.t. U is itself a vector space over $\mathbb{F}$, when equipped with the vector addition and scalar multiplication operations "inherited" from V.[a]

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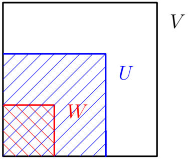
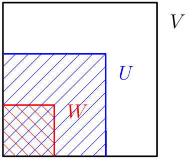
- This means that we add two vectors of U using the vector addition operation from V, and similar for scalar multiplication.
- Moreover, U must be "closed under" vector addition and scalar multiplication from V, that is, that the following hold:
  - $\forall \mathbf{u}_1, \mathbf{u}_2 \in U$: $\mathbf{u}_1 + \mathbf{u}_2 \in U$,
  - $\forall \mathbf{u} \in U$, $\alpha \in \mathbb{F}$: $\alpha \mathbf{u} \in U$,

  where vector addition and scalar multiplication are those from the vector space V.



- **Remark:** It is obvious that the subspace relation is transitive.
    - More precisely, for any vector space V over a field $\mathbb{F}$, if U is a subspace of V, and W is a subspace of U, then W is a subspace of V.



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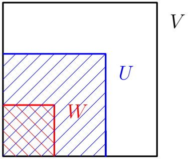

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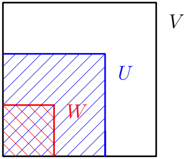

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Example 3.1.4

Let V be a vector space over a field $\mathbb{F}$. Then V is a subspace of itself, and $\{\mathbf{0}\}$ is a subspace of V.

- **Terminology:** For a vector space V over a field $\mathbb{F}$, the *trivial subspace* of V is the subspace $\{\mathbf{0}\}$. A *non-trivial* subspace of V is one that contains at least one non-zero vector. A subspace U of V is *proper* if $U \subsetneq V$.

Example 3.1.5

Let n be a positive integer, and let $\mathbb{F}$ be a field. Then $\mathbb{P}_{\mathbb{F}}^{n}$ is a subspace of $\mathbb{P}_{\mathbb{F}}$.

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- If you have studied calculus, here is another example.

Example 3.1.6

The real vector space of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ is a subspace of the real vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, which is in turn a subspace of the real vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

- Reminder:

Theorem 2.2.9

Let $(G, \circ)$ be a group with identity element e, and with the inverse of an element $a \in G$ denoted by a^{-1}. Then for all $H \subseteq G$, we have that $(H, \circ)$ is a subgroup of $(G, \circ)$ iff all the following hold:

1. (i) $e \in H$;
2. (ii) H is closed under $\circ$, that is, $\forall a, b \in H$: $a \circ b \in H$;
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- Theorem 3.1.7 (next slide) is an analog of Theorem 2.2.9 for vector (sub)spaces.

Theorem 3.1.7

Let V be a vector space over a field $\mathbb{F}$, and let $U \subseteq V$. Then U is a subspace of V iff the following three conditions are satisfied:

1. (i) $\mathbf{0} \in U$;[a]
2. (ii) U is closed under vector addition, $\forall \mathbf{u}, \mathbf{v} \in U$: $\mathbf{u} + \mathbf{v} \in U$;
3. (iii) U is closed under scalar multiplication, that is, $\forall \mathbf{u} \in U$, $\alpha \in \mathbb{F}$: $\alpha \mathbf{u} \in U$.

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Next, U satisfies axioms 2-5 from the definition of a vector space because the vector space V satisfies those axioms and because the vector addition and scalar multiplication operations in U are inherited from V.

Proof (continued). It remains to show that U satisfies axiom 1 from the definition of a vector space, that is, that U is an abelian group under vector addition. Since $(V, +)$ is an abelian group (because V is a vector space), it suffices to show that $(U, +)$ is a subgroup of $(V, +)$.

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By (i), we have that $\mathbf{0} \in U$, and by (ii), we have that U is closed under vector addition. Moreover, by (iii) and by Proposition 3.1.3(d), for all $\mathbf{u} \in U$, we have that $-\mathbf{u} = (-1)\mathbf{u} \in U$,

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By now adding $-\mathbf{0}_U$ to both sides of the equation (where $-\mathbf{0}_U$ is the additive inverse of the vector $\mathbf{0}_U$ in V), and we obtain $\mathbf{0}_U = \mathbf{0}$. So, (i) holds. $\square$

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Definition

Suppose that V is a vector space over a field $\mathbb{F}$. Given vectors $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$, we say that a vector $\mathbf{v} \in V$ is a *linear combination* of $\mathbf{u}_1, \dots, \mathbf{u}_k$ if there exist scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ s.t.

$$\mathbf{v} \quad = \quad \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k.$$

The *linear span* (or simply *span*) of the set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, denoted by $\text{Span}(\{\mathbf{u}_1, \dots, \mathbf{u}_k\})$ or $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$, is the set of all linear combinations of $\mathbf{u}_1, \dots, \mathbf{u}_k$, i.e.

$$\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k) \quad = \quad \{\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k \mid \alpha_1, \dots, \alpha_k \in \mathbb{F}\}.$$

As a convention, we define the "empty sum" of vectors in V to be $\mathbf{0}$ (the zero vector in V),[a] and consequently, $\text{Span}(\emptyset) = \{\mathbf{0}\}$.

[a] An "empty sum" might be the sum $\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$, where $k = 0$ (and so we do not actually have any $\mathbf{u}_i$'s or α_i's).

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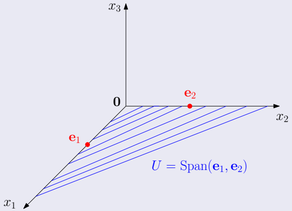
- **Terminology:** Given a vector space V over a field $\mathbb{F}$, and given vectors $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$, we say that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a *spanning set* of V, or that that the set $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ *spans* V, or that vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ *span* V, provided that $V = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$.
    - Note that $\emptyset$ is a spanning set of the trivial vector space $\{\mathbf{0}\}$ over a field $\mathbb{F}$.

Example 3.1.8

Consider vectors $\mathbf{e}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $\mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ in $\mathbb{R}^3$.

Then $\text{Span}(\mathbf{e}_1, \mathbf{e}_2) = \{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R}\}$. So, $\text{Span}(\mathbf{e}_1, \mathbf{e}_2)$ is the x_1x_2-plane in the Euclidean space $\mathbb{R}^3$.



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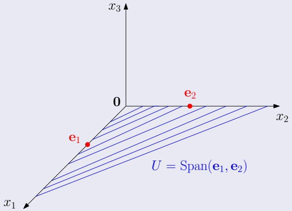

Example 3.1.9

Consider the polynomials $1, x, x^2$ in $\mathbb{P}_{\mathbb{R}}$. Then
$\text{Span}(1, x, x^2) = \{a_2 x^2 + a_1 x + a_0 \mid a_0, a_1, a_2 \in \mathbb{R}\} = \mathbb{P}^2_{\mathbb{R}}$.

- For a field $\mathbb{F}$ and a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ in $\mathbb{F}^{n \times m}$, we have that $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\}$.

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- We proved this in chapter 1, but here is the argument again:

$$\begin{aligned}
\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) &= \left\{x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m \mid x_1, \dots, x_m \in \mathbb{F}\right\} \\
&= \left\{\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \mid x_1, \dots, x_m \in \mathbb{F}\right\} \\
&= \left\{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\right\}.
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$$\begin{aligned} \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) &= \Big\{x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m \mid x_1, \dots, x_m \in \mathbb{F}\Big\} \\ &= \left\{\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \mid x_1, \dots, x_m \in \mathbb{F}\right\} \\ &= \Big\{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\Big\}. \end{aligned}$$

- Consequently, $\forall \mathbf{b} \in \mathbb{F}^n$, we have that $\mathbf{b} \in \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$ iff the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Proposition 3.1.10

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ ($m \geq 1$) be some vectors in $\mathbb{F}^n$. Set $A := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$. Then the following are equivalent:

(a) vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ span $\mathbb{F}^n$;

(b) for all $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent;

(c) $\text{rank}(A) = n$ (i.e. A has full row rank).

Proof.

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Proof. By Corollary 1.6.6, (b) and (c) are equivalent.

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Proof. By Corollary 1.6.6, (b) and (c) are equivalent.

On the other hand, the fact that (a) and (b) are equivalent essentially follows from the fact that

$$\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \quad = \quad \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\}.$$

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Proof. By Corollary 1.6.6, (b) and (c) are equivalent.

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$$\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \;=\; \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\}.$$

Indeed, we have the following sequence of equivalent statements (next slide):

Proof (continued).

$$\underbrace{\text{vectors } \mathbf{a}_1, \dots, \mathbf{a}_m \text{ span } \mathbb{F}^n}_{\text{(a)}} \iff \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \mathbb{F}^n$$

$$\iff \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m\} = \mathbb{F}^n$$

$$\iff \forall \mathbf{b} \in \mathbb{F}^n \ \exists \mathbf{x} \in \mathbb{F}^m \text{ s.t. } A\mathbf{x} = \mathbf{b}$$

$$\iff \underbrace{\forall \mathbf{b} \in \mathbb{F}^n\text{: } A\mathbf{x} = \mathbf{b} \text{ is consistent.}}_{(b)}$$

Thus, (a) and (b) are indeed equivalent. This completes the argument. □

Proposition 3.1.10

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ ($m \geq 1$) be some vectors in $\mathbb{F}^n$. Set $A := \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array}\right]$. Then the following are equivalent:

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Theorem 3.1.11

Let V be a vector space over a field $\mathbb{F}$, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$ $(k \geq 0)$.[a] Then all the following hold:

(a) $\mathbf{u}_1, \dots, \mathbf{u}_k \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$;

(b) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of V;

(c) for all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of U;

(d) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is precisely the intersection of all subspaces of V that contain the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$.

[a] If $k = 0$, then $\mathbf{u}_1, \dots, \mathbf{u}_k$ is an empty list of vectors, the set $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is empty, and $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k) = \{\mathbf{0}\}$.

Proof.

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Proof. To prove (a), we simply observe that for all $i \in \{1, \dots, k\}$, we have that

$$\mathbf{u}_i = 0\mathbf{u}_1 + \dots + 0\mathbf{u}_{i-1} + 1\mathbf{u}_i + 0\mathbf{u}_{i+1} + \dots + 0\mathbf{u}_k,$$

and so $\mathbf{u}_i \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$.

Proof (continued). Next, we prove (b).

(b) $\text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ is a subspace of V

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It suffices to show that $\mathit{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ satisfies (i), (ii) and (iii) from Theorem 3.1.7, that is, that all the following hold:

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For (i), we note that $\mathbf{0} = 0\mathbf{u}_1 + \dots + 0\mathbf{u}_k \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$.

Proof (continued). Next, we prove (b).

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Proof (continued). Next, we prove (b).

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Proof (continued). Next, we prove (b).

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and we deduce that $\mathbf{v}_1 + \mathbf{v}_2 \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. This proves (ii). The proof of (iii) is similar (details: Lecture Notes). This proves (b).

Proof (continued).

Claim. *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that $Span(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

Proof of the Claim.

Proof (continued).

> ***Claim.*** *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

Proof of the Claim. Fix any subspace U of V that contains $\mathbf{u}_1, \dots, \mathbf{u}_k$; WTS Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.

Proof (continued).

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Proof of the Claim. Fix any subspace U of V that contains $\mathbf{u}_1, \dots, \mathbf{u}_k$; WTS Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.

Fix any $\mathbf{v} \in$ Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then there exist scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ s.t.

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$$

Proof (continued).

> ***Claim.*** *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

Proof of the Claim. Fix any subspace U of V that contains $\mathbf{u}_1, \dots, \mathbf{u}_k$; WTS Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.

Fix any $\mathbf{v} \in$ Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then there exist scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ s.t.

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$$

Since U is a subspace of V, it satisfies (ii) and (iii) from Theorem 3.1.7.

Proof (continued).

> ***Claim.*** *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

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Proof (continued).

Claim. *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

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So, Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$. ♦

Proof (continued).

Claim. *For all subspaces U of V s.t.* $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$*, we have that Span*$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.

Proof (continued).

Claim. *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

We now prove (c).

(c) for all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of U

Proof (continued).

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Fix any subspace U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$. By the Claim, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$, and by (a), Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of V.

Proof (continued).

> ***Claim.*** *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

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Fix any subspace U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$. By the Claim, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$, and by (a), Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of V. So, U is a vector space, and Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is subset of U that is a vector space in its own right (when equipped with the vector addition and scalar multiplication operations inherited from U).

Proof (continued).

> ***Claim.*** *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

We now prove (c).

- (c) for all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of U

Fix any subspace U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$. By the Claim, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$, and by (a), Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of V. So, U is a vector space, and Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is subset of U that is a vector space in its own right (when equipped with the vector addition and scalar multiplication operations inherited from U). So, by definition, Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of U. This proves (c).

Proof (continued).

Claim. *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that $Span(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

Proof (continued).

> ***Claim.*** *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

It remains to prove (d).

(d) Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is precisely the intersection of all subspaces of V that contain the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$.

Proof (continued).

> ***Claim.*** *For all subspaces U of V s.t.* $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$*, we have that Span*$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.

It remains to prove (d).

(d) Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is precisely the intersection of all subspaces of V that contain the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$.

By (a) and (b), Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is itself a subspace of V that contains $\mathbf{u}_1, \dots, \mathbf{u}_k$.

Proof (continued).

> ***Claim.*** *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

It remains to prove (d).

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By (a) and (b), Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is itself a subspace of V that contains $\mathbf{u}_1, \dots, \mathbf{u}_k$. So, the intersection of all subspaces of V that contain $\mathbf{u}_1, \dots, \mathbf{u}_k$ is a subset of Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$.

Proof (continued).

> ***Claim.*** *For all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we have that Span$(\mathbf{u}_1, \dots, \mathbf{u}_k) \subseteq U$.*

It remains to prove (d).

(d) Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is precisely the intersection of all subspaces of V that contain the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$.

By (a) and (b), Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is itself a subspace of V that contains $\mathbf{u}_1, \dots, \mathbf{u}_k$. So, the intersection of all subspaces of V that contain $\mathbf{u}_1, \dots, \mathbf{u}_k$ is a subset of Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$.

On the other hand, by the Claim, Span$(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subset of each subspace of V that contains the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$, and consequently, of the intersection of all such subspaces. This proves (d). $\square$

Theorem 3.1.11

Let V be a vector space over a field $\mathbb{F}$, and let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$ $(k \geq 0)$.[a] Then all the following hold:

(a) $\mathbf{u}_1, \dots, \mathbf{u}_k \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$;

(b) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of V;

(c) for all subspaces U of V s.t. $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a subspace of U;

(d) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is precisely the intersection of all subspaces of V that contain the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$.

[a] If $k = 0$, then $\mathbf{u}_1, \dots, \mathbf{u}_k$ is an empty list of vectors, the set $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is empty, and $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k) = \{\mathbf{0}\}$.

- **Remark:** In some texts, for a vector space V over a field $\mathbb{F}$, and for vectors $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$, the linear span (or simply span) of $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is defined to be the intersection of all subspaces of V that contain $\mathbf{u}_1, \dots, \mathbf{u}_k$.
  - By Theorem 3.1.11, this definition is equivalent to the one that we gave at the beginning of this subsection.

Proposition 3.1.12

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \text{Span}(\alpha_1 \mathbf{v}_1, \dots, \alpha_k \mathbf{v}_k).$$

Proof.

Proposition 3.1.12

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$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \; = \; \text{Span}(\alpha_1 \mathbf{v}_1, \dots, \alpha_k \mathbf{v}_k).$$

Proof. We need to prove two inclusions:

1. $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \subseteq \text{Span}(\alpha_1 \mathbf{v}_1, \dots, \alpha_k \mathbf{v}_k)$;
2. $\text{Span}(\alpha_1 \mathbf{v}_1, \dots, \alpha_k \mathbf{v}_k) \subseteq \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

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Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \ = \ \text{Span}(\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k).$$

Proof. We need to prove two inclusions:

(i) $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \subseteq \text{Span}(\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k)$;

(ii) $\text{Span}(\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k) \subseteq \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

We prove (i); the proof of (ii) is similar and is left as an exercise.

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Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

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2. (ii) $\text{Span}(\alpha_1 \mathbf{v}_1, \dots, \alpha_k \mathbf{v}_k) \subseteq \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

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Fix any vector $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

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1. (i) $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \subseteq \text{Span}(\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k)$;
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We prove (i); the proof of (ii) is similar and is left as an exercise.

Fix any vector $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Then, by definition, there exist scalars $\beta_1, \dots, \beta_k \in \mathbb{F}$ s.t.

$$\mathbf{v} \;=\; \beta_1\mathbf{v}_1 + \dots + \beta_k\mathbf{v}_k.$$

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Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \text{Span}(\alpha_1 \mathbf{v}_1, \dots, \alpha_k \mathbf{v}_k).$$

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$$\mathbf{v} = \beta_1 \mathbf{v}_1 + \dots + \beta_k \mathbf{v}_k.$$

Since scalars $\alpha_1, \dots, \alpha_k$ are all non-zero, they have multiplicative inverses $\alpha_1^{-1}, \dots, \alpha_k^{-1}$, respectively.

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Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

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$$\mathbf{v} = \beta_1\mathbf{v}_1 + \dots + \beta_k\mathbf{v}_k.$$

Since scalars $\alpha_1, \dots, \alpha_k$ are all non-zero, they have multiplicative inverses $\alpha_1^{-1}, \dots, \alpha_k^{-1}$, respectively. We now have that

$$\mathbf{v} = \beta_1\mathbf{v}_1 + \dots + \beta_k\mathbf{v}_k = (\beta_1\alpha_1^{-1})(\alpha_1\mathbf{v}_1) + \dots + (\beta_k\alpha_k^{-1})(\alpha_k\mathbf{v}_k),$$

Proposition 3.1.12

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

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We prove (i); the proof of (ii) is similar and is left as an exercise. Fix any vector $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Then, by definition, there exist scalars $\beta_1, \dots, \beta_k \in \mathbb{F}$ s.t.

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Since scalars $\alpha_1, \dots, \alpha_k$ are all non-zero, they have multiplicative inverses $\alpha_1^{-1}, \dots, \alpha_k^{-1}$, respectively. We now have that

$$\mathbf{v} \; = \; \beta_1 \mathbf{v}_1 + \dots + \beta_k \mathbf{v}_k \; = \; (\beta_1 \alpha_1^{-1})(\alpha_1 \mathbf{v}_1) + \dots + (\beta_k \alpha_k^{-1})(\alpha_k \mathbf{v}_k),$$

and so $\mathbf{v} \in \text{Span}(\alpha_1 \mathbf{v}_1, \dots, \alpha_k \mathbf{v}_k)$.

Proposition 3.1.12

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \text{Span}(\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k).$$

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(i) $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \subseteq \text{Span}(\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k)$;

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We prove (i); the proof of (ii) is similar and is left as an exercise. Fix any vector $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Then, by definition, there exist scalars $\beta_1, \dots, \beta_k \in \mathbb{F}$ s.t.

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Since scalars $\alpha_1, \dots, \alpha_k$ are all non-zero, they have multiplicative inverses $\alpha_1^{-1}, \dots, \alpha_k^{-1}$, respectively. We now have that

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and so $\mathbf{v} \in \text{Span}(\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k)$. This proves (i). $\square$

Proposition 3.1.12

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

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- **Remark:** In Proposition 3.1.12, it is important that the scalars $\alpha_1, \dots, \alpha_k$ are all non-zero, and indeed, the proposition becomes false without this hypothesis.

Proposition 3.1.12

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \;=\; \text{Span}(\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k).$$

- **Remark:** In Proposition 3.1.12, it is important that the scalars $\alpha_1, \dots, \alpha_k$ are all non-zero, and indeed, the proposition becomes false without this hypothesis.
  - For example, for the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ in $\mathbb{R}^2$, we have that $\text{Span}(\mathbf{e}_1, \mathbf{e}_2) = \mathbb{R}^2$, but

$$\text{Span}(1\mathbf{e}_1, 0\mathbf{e}_2) = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \mid x_1 \in \mathbb{R} \right\},$$

which is a proper subspace of $\mathbb{R}^2$.

- Suppose we are given two vector spaces, say U and W, over a field $\mathbb{F}$.

- Suppose we are given two vector spaces, say U and W, over a field $\mathbb{F}$.
- Then the **Cartesian product**

$$U \times W \quad := \quad \{(\mathbf{u}, \mathbf{w}) \mid \mathbf{u} \in U,\ \mathbf{w} \in W\}$$

can be turned into a vector space over $\mathbb{F}$ in a natural way.

- Suppose we are given two vector spaces, say U and W, over a field $\mathbb{F}$.
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$$U \times W \quad := \quad \{(\mathbf{u}, \mathbf{w}) \mid \mathbf{u} \in U,\ \mathbf{w} \in W\}$$

can be turned into a vector space over $\mathbb{F}$ in a natural way.
- We define vector addition in $U \times W$ by setting

$$(\mathbf{u}_1, \mathbf{w}_1) + (\mathbf{u}_2, \mathbf{w}_2) \quad := \quad (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2),$$

for all $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$, where in the first coordinate ("$\mathbf{u}_1 + \mathbf{u}_2$") we applied addition from the vector space U, and in the second coordinate ("$\mathbf{w}_1 + \mathbf{w}_2$") we applied vector addition from the vector space W.

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- Scalar multiplication in $U \times W$ (with scalars from the field $\mathbb{F}$) is defined in an equally natural way, i.e. by setting

$$\alpha(\mathbf{u}, \mathbf{w}) \quad := \quad (\alpha\mathbf{u}, \alpha\mathbf{w})$$

  for all $\alpha \in \mathbb{F}$, $\mathbf{u} \in U$, and $\mathbf{w} \in W$.

- The zero vector of $U \times W$ is the vector

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  where $-\mathbf{u}$ (resp. $-\mathbf{w}$) is the additive inverse of $\mathbf{u}$ (resp. $\mathbf{w}$) in the vector space U (resp. W).
- It is straightforward to verify that all the axioms of a vector space hold for $U \times W$ (with vector addition and scalar multiplication defined as above).
  - Indeed, this simply follows from the fact that those axioms hold for U and W, and the details are left as an exercise.

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- The details are left as an exercise.

Definition

Given a vector space V over a field $\mathbb{F}$, and given vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, we say that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a *linearly independent set*, or that vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are *linearly independent*, if for all scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ s.t.

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0},$$

we have that $\alpha_1 = \cdots = \alpha_k = 0$. In other words, vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent iff the equation $\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0}$ has only the "trivial solution," i.e. the solution $\alpha_1 = \cdots = \alpha_k = 0$. On the other hand, if vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are not linearly independent, then we say that they are *linearly dependent*, or that the $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a *linearly dependent set*.

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- So, vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent iff there exist scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$, not all zero, s.t. $\alpha_1 \mathbf{v}_1 + \dots + \alpha_k \mathbf{v}_k = \mathbf{0}$.

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- So, vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent iff there exist scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$, not all zero, s.t. $\alpha_1 \mathbf{v}_1 + \dots + \alpha_k \mathbf{v}_k = \mathbf{0}$.
- We note that $\emptyset$ is a linearly independent set in **any** vector space.

- For the special case of $\mathbb{F}^n$ (where $\mathbb{F}$ is a field), we have the following proposition.

Proposition 3.2.1

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ ($m \geq 1$) be vectors in $\mathbb{F}^n$. Set $A := \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$. Then the following are equivalent:

(a) vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ are linearly independent;

(b) the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (i.e. the solution $\mathbf{x} = \mathbf{0}$);

(c) $\text{rank}(A) = m$ (i.e. A has full column rank).

Proof.

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Proof. By Corollary 1.6.5, (b) and (c) are equivalent.

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Proof. By Corollary 1.6.5, (b) and (c) are equivalent.

Let us show that (a) and (b) are equivalent. We have the following sequence of equivalent statements (new slide):

Proof (continued).

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$$\underbrace{\text{vectors } \mathbf{a}_1, \dots, \mathbf{a}_m \text{ are linearly independent}}_{\text{(a)}}$$

$\iff$ the equation $x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m = \mathbf{0}$ has only the trivial solutuon (i.e. the solution $x_1 = \cdots = x_m = 0$)

$\iff$ the equation $\underbrace{\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}}_{=A} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \mathbf{0}$ has only the trivial solutuon (i.e. the solution $x_1 = \cdots = x_m = 0$)

$\iff$ $\underbrace{\text{the homogeneous matrix-vector equation } A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution (i.e. the solution } \mathbf{x} = \mathbf{0}\text{).}}_{\text{(b)}}$

Thus, (a) and (b) are equivalent. This completes the argument. $\square$

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Proposition 3.2.2

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent iff the set $\{\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k\}$ is linearly independent.

Proof. This readily follows from the definition of linear independence and is left as an exercise. □

Definition

A *finite basis* (or simply *basis*) of a vector space V over a field $\mathbb{F}$ is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors in V that satisfies the following two conditions:

1. $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent in V;
2. $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a spanning set of V, i.e. $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = V$.

Example 3.2.3

Let $\mathbb{F}$ be a field. Then the standard basis $\mathcal{E}_n = \{\mathbf{e}_1^n, \dots, \mathbf{e}_n^n\}$ of $\mathbb{F}^n$ is indeed a basis of $\mathbb{F}^n$.

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Example 3.2.4

Let $\mathbb{F}$ be a field. Then

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of $\mathbb{F}^{3\times 2}$.

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Definition

A vector space is *finite-dimensional* if it has a finite basis. A vector space that does not have a finite basis is *infinite-dimensional*.

Proposition 3.2.5

Let $\mathbb{F}$ be a field. Then $\mathbb{P}_{\mathbb{F}}$ is inifnite-dimensional. On the other hand, for all non-negative integers n, $\{1, x, \dots, x^n\}$ is a basis of $\mathbb{P}^n_{\mathbb{F}}$, and in particular, $\mathbb{P}^n_{\mathbb{F}}$ is finite-dimensional.

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Any finite set $\mathcal{P} = \{p_1(x), \ldots, p_k(x)\}$ of polynomials in $\mathbb{P}_{\mathbb{F}}$ has bounded degree (say, at most d), and so x^{d+1} is not in Span($\mathcal{P}$).

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Any finite set $\mathcal{P} = \{p_1(x), \dots, p_k(x)\}$ of polynomials in $\mathbb{P}_{\mathbb{F}}$ has bounded degree (say, at most d), and so x^{d+1} is not in Span$(\mathcal{P})$. So, $\mathbb{P}_{\mathbb{F}}$ does not have a finite spanning set, and consequently, it does not have a finite basis. $\square$

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- It is, indeed, possible to define a basis more generally, so that it may possibly be an infinite set.
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- **Notation:** Suppose that V is a vector-space over a field $\mathbb{F}$.
  - If V is finite-dimensional (i.e. has a finite basis), then we write $\dim(V) < \infty$.
  - On the other hand, if V is infinite-dimensional (i.e. does not have a finite basis), then we write $\dim(V) = \infty$.

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    - $\emptyset$ is a basis of the trivial vector space $\{\mathbf{0}\}$ (over any field $\mathbb{F}$), and in particular, $\{\mathbf{0}\}$ is finite dimensional.
        - In fact, $\emptyset$ is the unique basis of $\{\mathbf{0}\}$ (because, by the previous bullet point, no linearly independent set contains $\mathbf{0}$).

- **Remarks:** Suppose that V is a vector space over a field $\mathbb{F}$.
  - Suppose we are given vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and we are trying to check if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a spanning set of V, i.e. whether $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ (this is one of the two conditions from the definition of a basis).

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    - Obviously, $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \subseteq V$, and so the only question is whether $V \subseteq \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

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    - Obviously, $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) \subseteq V$, and so the only question is whether $V \subseteq \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.
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    - So, the second condition from the definition of a basis holds iff every vector in V is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$.

- In the literature, there is a bit of ambiguity about whether (finite) bases are sets or **ordered** sets.
  - An "ordered set" is a set in which order and repetitions matter.
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- It is important to note that no linearly independent set (and in particular, no basis), may contain more than one copy of the same vector.
  - Indeed, if $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a list of vectors that contains more than one copy of some vector (say, $\mathbf{v}_i = \mathbf{v}_j$ for some $i \neq j$), then we can set $\alpha_i = 1$, $\alpha_j = -1$, and $\alpha_k = 0$ for all $k \in \{1, \dots, n\} \setminus \{i, j\}$, and we get $\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0}$; so $\mathbf{v}_1, \dots, \mathbf{v}_n$ are not linearly independent.

- In what follows, if A and B are ordered sets (possibly with repeating elements), then $A \subseteq B$ means that every element of A appears at least as many times in B as in A.

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- Moreover, for $x \in A$, $A \setminus \{x\}$ is the set obtained from A by deleting one copy of x.

- For the special case of $\mathbb{F}^n$ (where $\mathbb{F}$ is a field), we have the following proposition.

Proposition 3.2.6

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ $(m \geq 1)$ be vectors in $\mathbb{F}^n$. Set $A := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$. Then $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ is a basis of $\mathbb{F}^n$ iff $\text{rank}(A) = n = m$ (i.e. A is a square matrix of full rank). In particular, every basis of $\mathbb{F}^n$ contains exactly n vectors.

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So, $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ is a basis of $\mathbb{F}^n$ iff $\text{rank}(A) = m = n$. $\square$

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    - So, Proposition 3.2.6 yields another characterizations of invertible matrices: a matrix in $\mathbb{F}^{n \times n}$ (where $\mathbb{F}$ is a field) is invertible iff its columns form a basis of $\mathbb{F}^n$.
  - By Proposition 3.2.6, every basis of $\mathbb{F}^n$ (where $\mathbb{F}$ is a field) contains exactly n vectors.
    - In fact (see Theorem 3.2.16), if V is **any** finite-dimensional vector space, then all bases of V are of the same size (i.e. contain exactly the same number of vectors).
    - However, to prove this, we first need to develop some more theory.

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Let V be a vector space over a field $\mathbb{F}$, and let $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V;

(ii) for all vectors $\mathbf{v} \in V$, there exist **unique** scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$.

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Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V, and consequently a spanning set of V, we know that every vector in V is a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. This proves existence.

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Proof (continued). It remains to prove uniqueness.

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By (ii), every vector in V is a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and so $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

It remains to show that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent. Clearly, the equation $\alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n = \mathbf{0}$ has a solution, namely $\alpha_1 = \dots = \alpha_n = 0$;

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in $\mathbb{F}^n$ such that $\mathbf{v} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n$; the vector $\left[\ \mathbf{v}\ \right]_{\mathcal{B}}$ is called the *coordinate vector* of $\mathbf{v}$ associated with the basis $\mathcal{B}$.

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  - So, V is in a sense "equivalent" to $\mathbb{F}^n$.
    - The technical term here is "isomorphic": V is "isomorphic" to $\mathbb{F}^n$.
    - We will discuss this more formally in chapter 4.

Example 3.2.8

Let $\mathbb{F}$ be a field.

(a) Consider the basis $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathbb{F}^n$. Then for all $\mathbf{x} \in \mathbb{F}^n$, we have that $\left[\ \mathbf{x}\ \right]_{\mathcal{E}_n} = \mathbf{x}$.[a]

(b) Consider the basis $\mathcal{B} := \{1, x, \dots, x^n\}$ of $\mathbb{P}^n_{\mathbb{F}}$. Then for all polynomials $p(x) = a_n x^n + \dots + a_1 x + a_0$ in $\mathbb{P}^n_{\mathbb{F}}$ (where $a_n, \dots, a_1, a_0 \in \mathbb{F}$), we have that $\left[\ p(x)\ \right]_{\mathcal{B}} = \left[\begin{array}{cccc} a_0 & a_1 & \dots & a_n \end{array}\right]^T$.

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[a] Indeed, for any $\mathbf{x} = \left[\begin{array}{ccc} x_1 & \dots & x_n \end{array}\right]^T$, we have that $\mathbf{x} = x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n$, and so $\left[\ \mathbf{x}\ \right]_{\mathcal{E}_n} = \left[\begin{array}{ccc} x_1 & \dots & x_n \end{array}\right]^T = \mathbf{x}$.

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- See the Lecture Notes for an example with matrices.

- **Remark:** When working with coordinate vectors, we must always **specify which basis we are working with**.

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  - This is because the same vector of a given finite-dimensional vector space may have different coordinate vectors with respect to different bases.
  - Changing the order of basis elements may change the coordinate vectors. This is, in fact, the main reason for treating bases as **ordered** sets, rather than simply sets.

Proposition 3.2.9

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ $(n \geq 1)$ be a basis of a vector space V over a field $\mathbb{F}$. Then for all $i \in \{1, \dots, n\}$, we have that $\left[\ \mathbf{b}_i \ \right]_{\mathcal{B}} = \mathbf{e}_i^n$.

Proof.

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Proof. Fix $i \in \{1, \dots, n\}$. Then

$$\mathbf{b}_i = 0\mathbf{b}_1 + \dots + 0\mathbf{b}_{i-1} + 1\mathbf{b}_i + 0\mathbf{b}_{i+1} + \dots + 0\mathbf{b}_n$$

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and consequently,

$$\left[\ \mathbf{b}_i\ \right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{-th entry}$$

i.e. $\left[\ \mathbf{b}_i\ \right]_{\mathcal{B}} = \mathbf{e}_i^n$. $\square$

- Reminder:

Proposition 3.1.12

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \text{Span}(\alpha_1 \mathbf{v}_1, \dots, \alpha_k \mathbf{v}_k).$$

Proposition 3.2.2

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent iff the set $\{\alpha_1 \mathbf{v}_1, \dots, \alpha_k \mathbf{v}_k\}$ is linearly independent.

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- Putting Propositions 3.1.12 and 3.2.2 together, we get the following result for bases (next slide):

Proposition 3.2.10

Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$, and let $\alpha_1, \ldots, \alpha_n \in \mathbb{F} \setminus \{0\}$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V if and only if $\{\alpha_1 \mathbf{v}_1, \ldots, \alpha_k \mathbf{v}_n\}$ is a basis of V.

Proof. This follows immediately from the definition of a basis and from Propositions 3.1.12 and 3.2.2. $\square$