Linear Algebra 1

Lecture #7

Permutations and the symmetric group. Fields

Irena Penev

November 20, 2023

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1. Permutations and the symmetric group

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2. Fields

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  - Moreover, it is clear that $\circ$ is associative; indeed, for any $\pi, \sigma, \tau \in \text{Sym}(X)$, we have that $\pi \circ (\sigma \circ \tau) = (\pi \circ \sigma) \circ \tau$, because for all $x \in X$, we have the following:

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  - The identity element of this group is the identity function Id_X.
  - The inverse element of any permutation $\pi \in \mathrm{Sym}(X)$ is the inverse permutation π^{-1}.

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- So, $\sigma \circ \tau \neq \tau \circ \sigma$.
- Thus, $\mathrm{Sym}(X)$ is not abelian.

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- The group $(S_n, \circ)$ is called the *symmetric group of degree n*.
- Note that $|S_n| = n!$.

- A permutation $\pi \in S_n$ can be represented in the following way:

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- For example, the permutation $\pi \in S_4$ given by
  - $\pi(1) = 3$,
  - $\pi(2) = 2$,
  - $\pi(3) = 4$,
  - $\pi(4) = 1$

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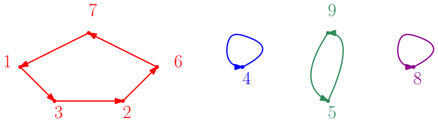

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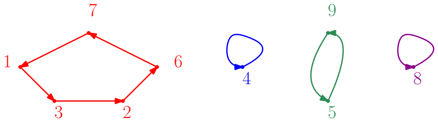

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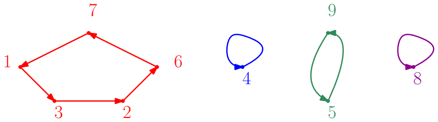

- We can "encode" the picture that we obtained as a "product of disjoint cycles":

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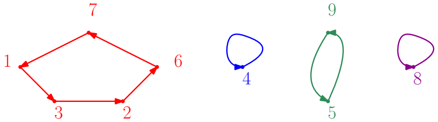
- The above is also referred to as a "disjoint cycle decomposition" of the permutation π.

- Reminder: $\pi = (13267)(4)(59)(8)$.



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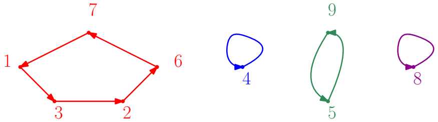

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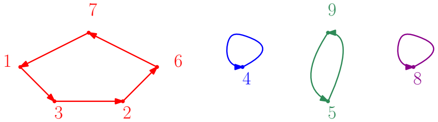

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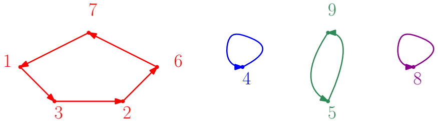

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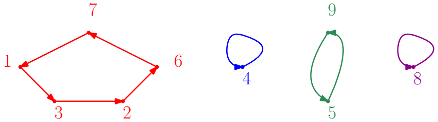

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- Usually, the canonical representation is preferred, but occasionally, it may be more practical to use a non-canonical one.

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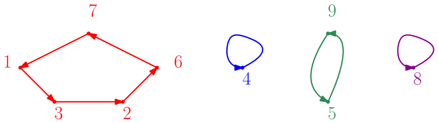

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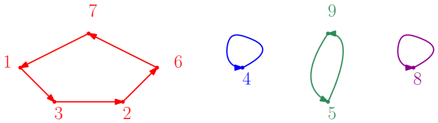

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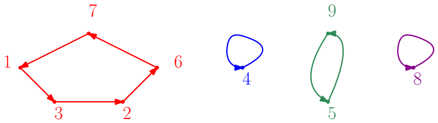

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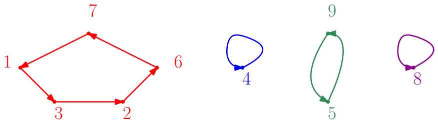

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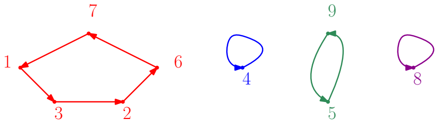

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- In this case, the cycles (4) and (8) are understood from context.
- However, we can only do this when n has been specified beforehand!
  - Otherwise, cycles of length one must be included.

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  - For instance, if we are working in S_{12}, then (123) is ambiguous.
  - To avoid ambiguity, we write $(1, 2, 3)$ or $(12, 3)$, as appropriate.
  - However, if we are working in S_n, where n is a single-digit number, then there is no danger of confusion, and so we normally omit commas.

Example 2.3.1

Find the disjoint cycle decompositions of the following permutations.

(a) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$

(b) $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix}$

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Solution. We have:

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(b) $\pi_2 = (134)(2)(56)$;

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Solution. We have:

(a) $\pi_1 = (125)(34)$;

(b) $\pi_2 = (134)(2)(56)$;

- we could also have written $\pi \in S_6$, $\pi = (134)(56)$;

Example 2.3.1

Find the disjoint cycle decompositions of the following permutations.

(a) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$

(b) $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix}$

(c) $\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}$

Solution. We have:

(a) $\pi_1 = (125)(34)$;

(b) $\pi_2 = (134)(2)(56)$;
  - we could also have written $\pi \in S_6$, $\pi = (134)(56)$;

(c) $\pi_3 = (12543)$.

□

- It is also easy to go the other way around: from the disjoint cycle decomposition to the table representation.

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- For instance:
  - $(143)(26)(5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 1 & 3 & 5 & 2 \end{pmatrix}$;

- It is also easy to go the other way around: from the disjoint cycle decomposition to the table representation.
- For instance:
  - $(143)(26)(5) = \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 1 & 3 & 5 & 2 \end{array} \right)$;
  - $(154362) = \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 3 & 4 & 2 \end{array} \right)$.

- By composing two permutations, we get another permutation.

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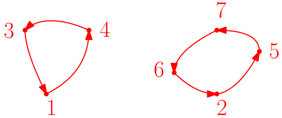
- For example:
  - $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix};$

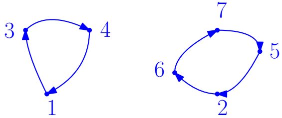
- By composing two permutations, we get another permutation.
- For example:
  - $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix};$
  - $(1)(23)(45) \circ (124)(35) = (134)(25).$

- The inverse of a permutation π in S_n can be obtained by starting with a disjoint cycle decomposition of π, and then reversing the order of elements in all cycles, i.e. turning each cycle of the form $(a_1 a_2 \ldots a_k)$ into $(a_k \ldots a_2 a_1)$.

- For example, in S_7:
  - if $\pi_1 = (143)(2576)$, then $\pi_1^{-1} = (341)(6752) = (134)(2675)$;

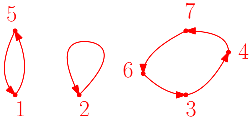


π_1

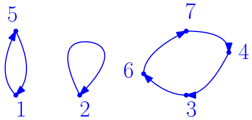


π_1^{-1}

  - if $\pi_2 = (15)(2)(3476)$, then
    $\pi_2^{-1} = (51)(2)(6743) = (15)(2)(3674)$.



π_2



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- So, in this context, we have that

$$1 \;=\; (1)(2)\ldots(n).$$

- If we wish to emphasize n (or if we need to avoid confusion with other kinds of 1 that may appear in our proof/computation), then we can denote the identity permutation in S_n by 1_n.

Definition

Given a positive integer n and a permutation $\pi \in S_n$, the *sign* of π, denoted by $\mathrm{sgn}(\pi)$, is given by $\mathrm{sgn}(\pi) = (-1)^{n-k}$, where k is the number of cycles in the disjoint cycle decomposition of π **including the one-element cycles**.

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- For instance:
  - for $\pi_1 = (1367)(2)(45)$ in S_7, we have
    $$\operatorname{sgn}(\pi_1) = (-1)^{7-3} = 1;$$
  - for $\pi_2 = (12)(345)(6)(7)$ in S_7, we have
    $$\operatorname{sgn}(\pi_2) = (-1)^{7-4} = -1.$$

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- Equivalently, for $\pi \in S_n$, we have that $\text{sgn}(\pi) = (-1)^{n'-k'}$, where k' is the number of cycles in some disjoint cycles in some disjoint cycle decomposition of π (possibly with some one-element cycles omitted), and n' is the number of elements in those k' cycles.

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- The two definitions are equivalent because if d is the number of omitted one-element cycles in some disjoint cycle decomposition of π , then $n = n' + d$, and if we write the complete disjoint cycle decomposition of π including all one-element cycles, then we get $k = k' + d$ many cycles.

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- The two definitions are equivalent because if d is the number of omitted one-element cycles in some disjoint cycle decomposition of π , then $n = n' + d$, and if we write the complete disjoint cycle decomposition of π including all one-element cycles, then we get $k = k' + d$ many cycles. So, $n - k = n' - k'$, and consequently, $(-1)^{n-k} = (-1)^{n'-k'}$.

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- Note that the one-element cycles (6) and (7) are implicitly understood for π_3, that is, $\pi_3 = (123)(45)(6)(7)$.
- And indeed, we have

$$\mathrm{sgn}(\pi_3) \;\; = \;\; (-1)^{7-4} = -1,$$

as before.

- **Remark:** Note that for all positive integers n, the identity permutation in S_n has sign 1.
    - This is because the identity permutation in S_n has disjoint cycle decomposition $(1)(2)\dots(n)$, and so its sign is $(-1)^{n-n} = (-1)^0 = 1$.

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- **Terminology:** Permutations whose sign is $+1$ are called *even*, and permutations whose sign is -1 are called *odd*. Since the sign of the identity permutation is $+1$, the identity permutation is even.

Proposition 2.3.2

Let $n \geq 2$ be an integer, and let π be a permutation in S_n. Then $\operatorname{sgn}(\pi^{-1}) = \operatorname{sgn}(\pi)$.

Proof.

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Proof. This follows from the fact that π and π^{-1} have the same number of cycles in their disjoint cycle decompositions (when the one-element cycles are included).

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  - $\pi(i) = j$,
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- For instance, the following permutation in S_5 is a transposition:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix} = (25).$$

- Note that this transposition could also have been written in the form $(1)(25)(3)(4)$.
  - More commonly, one-element cycles are omitted.

- **Remark:** Every transposition is its own inverse, that is, for any transposition $\tau = (ij)$ in S_n ($n \geq 2$), we have that $\tau^{-1} = \tau$.

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  - Indeed, if τ is a transposition in S_n $(n \geq 2)$, then the disjoint cycle decomposition of τ consists of one cycle of length two and $n - 2$ many cycles of length one, and consequently, it consists of $n - 1$ cycles total (when cycles of length one are included).

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  - So, $\mathrm{sgn}(\tau) = (-1)^{n-(n-1)} = -1$.

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  - The correctness of the above can easily be verified by checking that the image of each element of $\{1, \ldots, 7\}$ under the permutations $(134)(2657)$ and $(13) \circ (34) \circ (26) \circ (65) \circ (57)$ is the same.

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- Moreover, this works in general, as the following proposition shows (next slide).

Proposition 2.3.3

Let $n \geq 2$ be an integer. Then any permutation in S_n can be written as a composition of transpositions.

Proof.

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Let us now suppose that π is some permutation in S_n other than the identity. Then π can be written as the product of one or more disjoint cycles of length at least two (one-element cycles are omitted in our expression, but are understood from context).

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Let us now suppose that π is some permutation in S_n other than the identity. Then π can be written as the product of one or more disjoint cycles of length at least two (one-element cycles are omitted in our expression, but are understood from context). Let us say we have k cycles of length at least two:

$$\pi = (a_1^1 a_2^1 \ldots a_{\ell_1}^1) \ldots (a_1^k a_2^k \ldots a_{\ell_k}^k),$$

where the a_i^j's are pairwise distinct, and $\ell_1, \ldots, \ell_k \geq 2$.

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where the a_i^j's are pairwise distinct, and $\ell_1, \dots, \ell_k \geq 2$. But then

$$\pi = (a_1^1 a_2^1) \circ (a_2^1 a_3^1) \circ \cdots \circ (a_{\ell_1 - 1}^1 a_{\ell_1}^1) \circ \cdots \circ (a_1^k a_2^k) \circ (a_2^k a_3^k) \circ \cdots \circ (a_{\ell_k - 1}^k a_{\ell_k}^k),$$

and so π is the composition of transpositions. $\square$

Example 2.3.4

Express each of the following permutations in S_6 as the composition of transpositions.

(a) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 2 & 3 & 4 & 6 \end{pmatrix}$;

(b) $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \end{pmatrix}$;

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Solution.

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Solution.

(a) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 2 & 3 & 4 & 6 \end{pmatrix} = (2543) = (25) \circ (54) \circ (43)$;

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Solution.

(a) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 2 & 3 & 4 & 6 \end{pmatrix} = (2543) = (25) \circ (54) \circ (43)$;

(b) $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \end{pmatrix} = {\color{red}(12)}{\color{blue}(456)} = {\color{red}(12)} \circ {\color{blue}(45)} \circ {\color{blue}(56)}$;

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(c) $\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 2 & 1 & 4 \end{pmatrix}$.

Solution (continued).

(c) $\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 2 & 1 & 4 \end{pmatrix} = (135)(264) = (13) \circ (35) \circ (26) \circ (64)$.

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- We note that the same permutation can be expressed as the composition of transpositions in more than one way.

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- For instance, in S_5, we have:
  - $(12345) = (12) \circ (23) \circ (34) \circ (45)$;
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  - $(12345) = (15) \circ (14) \circ (13) \circ (12)$;
  - $(12345) =$ $(35) \circ (35) \circ (23) \circ (23) \circ (15) \circ (14) \circ (13) \circ (12) \circ (35) \circ (35)$.

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  - $(12345) =$ $(35) \circ (35) \circ (23) \circ (23) \circ (15) \circ (14) \circ (13) \circ (12) \circ (35) \circ (35)$.
- However, as we shall see, for any given permutation π in S_n, where $n \geq 2$, in all representations of π as a composition of transpositions, the number of transpositions is of the same parity (i.e. it is either always even or always odd).

Theorem 2.3.6

Let $n \geq 2$. Then for any permutation $\pi \in S_n$, if π can be expressed as a composition of r transpositions, then

a. $\mathrm{sgn}(\pi) = (-1)^r$;
b. π is an even permutation iff r is even;
c. π is an odd permutation iff r is odd.

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Proposition 2.3.5

Let $n \geq 2$ be an integer. Then for all $\pi, \tau \in S_n$ s.t. τ is a **transposition**, we have that $\text{sgn}(\tau \circ \pi) = \text{sgn}(\pi \circ \tau) = -\text{sgn}(\pi)$.

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- We first prove Theorem 2.3.6 assuming Proposition 2.3.5, and then we actually prove Proposition 2.3.5.

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Proof (assuming Proposition 2.3.5).

Proposition 2.3.5

Let $n \geq 2$ be an integer. Then for all $\pi, \tau \in S_n$ s.t. τ is a **transposition**, we have that $\mathrm{sgn}(\tau \circ \pi) = \mathrm{sgn}(\pi \circ \tau) = -\mathrm{sgn}(\pi)$.

Theorem 2.3.6

Let $n \geq 2$. Then for any permutation $\pi \in S_n$, if π can be expressed as a composition of r transpositions, then

(a) $\mathrm{sgn}(\pi) = (-1)^r$;

(b) π is an even permutation iff r is even;

(c) π is an odd permutation iff r is odd.

Proof (assuming Proposition 2.3.5). Clearly, (b) and (c) follow from (a).

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Let $n \geq 2$ be an integer. Then for all $\pi, \tau \in S_n$ s.t. τ is a **transposition**, we have that $\mathrm{sgn}(\tau \circ \pi) = \mathrm{sgn}(\pi \circ \tau) = -\mathrm{sgn}(\pi)$.

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Proof (assuming Proposition 2.3.5). Clearly, (b) and (c) follow from (a). Part (a) follows from Proposition 2.3.5 by an easy induction on r. Let us give the details. We prove the following statement: "for every positive integer r and permutation $\pi \in S_n$, if π is the composition of r transpositions, then $\text{sgn}(\pi) = (-1)^r$."

Proof (continued). Reminder: WTS for every positive integer r and permutation $\pi \in S_n$, if π is the composition of r transpositions, then $\mathrm{sgn}(\pi) = (-1)^r$.

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Base case: $r = 1$. Note that if π is the composition of one transposition, i.e. π is itself a transposition, then π is odd, and we have that $\operatorname{sgn}(\pi) = -1 = (-1)^r$.

Proof (continued). Reminder: WTS for every positive integer r and permutation $\pi \in S_n$, if π is the composition of r transpositions, then $\operatorname{sgn}(\pi) = (-1)^r$.

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Induction step: Fix a positive integer r, and assume that for any permutation $\pi \in S_n$, if π is the composition of r transpositions, then $\operatorname{sgn}(\pi) = (-1)^r$.

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Now, fix a permutation $\pi \in S_n$ in S_n s.t. π can be expressed as the composition of $r + 1$ transpositions, say
$\pi = (a_0 a_0') \circ (a_1 a_1') \circ \cdots \circ (a_r a_r')$.

Proof (continued). Reminder: WTS for every positive integer r and permutation $\pi \in S_n$, if π is the composition of r transpositions, then $\text{sgn}(\pi) = (-1)^r$.

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Then by the induction hypothesis, $\pi' := (a_1 a_1') \circ \cdots \circ (a_r a_r')$ satisfies $\text{sgn}(\pi') = (-1)^r$.

Proof (continued). Reminder: WTS for every positive integer r and permutation $\pi \in S_n$, if π is the composition of r transpositions, then $\operatorname{sgn}(\pi) = (-1)^r$.

Base case: $r = 1$. Note that if π is the composition of one transposition, i.e. π is itself a transposition, then π is odd, and we have that $\operatorname{sgn}(\pi) = -1 = (-1)^r$.

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Proof (continued). Reminder: WTS for every positive integer r and permutation $\pi \in S_n$, if π is the composition of r transpositions, then $\operatorname{sgn}(\pi) = (-1)^r$.

Base case: $r = 1$. Note that if π is the composition of one transposition, i.e. π is itself a transposition, then π is odd, and we have that $\operatorname{sgn}(\pi) = -1 = (-1)^r$.

Induction step: Fix a positive integer r, and assume that for any permutation $\pi \in S_n$, if π is the composition of r transpositions, then $\operatorname{sgn}(\pi) = (-1)^r$.

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Then by the induction hypothesis, $\pi' := (a_1 a_1') \circ \cdots \circ (a_r a_r')$ satisfies $\operatorname{sgn}(\pi') = (-1)^r$. But since $\pi = (a_0 a_0') \circ \pi'$, Proposition 2.3.5 guarantees that $\operatorname{sgn}(\pi) = -\operatorname{sgn}(\pi')$. So, $\operatorname{sgn}(\pi) = -\operatorname{sgn}(\pi') = -(-1)^r = (-1)^{r+1}$. This completes the induction. $\square$

Proposition 2.3.5

Let $n \geq 2$ be an integer. Then for all $\pi, \tau \in S_n$ s.t. τ is a **transposition**, we have that $\mathrm{sgn}(\tau \circ \pi) = \mathrm{sgn}(\pi \circ \tau) = -\mathrm{sgn}(\pi)$.

- **Warning:** In general, $\tau \circ \pi \neq \pi \circ \tau$.

Proof of Proposition 2.3.5.

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Let $n \geq 2$ be an integer. Then for all $\pi, \tau \in S_n$ s.t. τ is a **transposition**, we have that $\text{sgn}(\tau \circ \pi) = \text{sgn}(\pi \circ \tau) = -\text{sgn}(\pi)$.

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Proof of Proposition 2.3.5. The Claim below proves one part of the proposition ("$\text{sgn}(\tau \circ \pi) = -\text{sgn}(\pi)$"). The other part ("$\text{sgn}(\pi \circ \tau) = -\text{sgn}(\pi)$") can be proven using the Claim and certain basic properties of permutations (as we shall see below).

> ***Claim.*** *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

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Proof of the Claim.

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> ***Claim.*** *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

Proof of the Claim. Fix $\pi, \tau \in S_n$, and assume that $\tau = (ij)$ is a transposition (here, i and j are some two distinct elements of $\{1, \dots, n\}$). There are two cases to consider: when i and j are in the same cycle of the disjoint cycle decomposition of π, and when they are in different cycles.

Claim. *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

Proof of the Claim (continued). Reminder: $\tau = (ij)$.

Claim. *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

Proof of the Claim (continued). Reminder: $\tau = (ij)$.

Case 1: i and j are in the same cycle of the disjoint cycle decomposition of π.

Claim. *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

Proof of the Claim (continued). Reminder: $\tau = (ij)$.

Case 1: i and j are in the same cycle of the disjoint cycle decomposition of π. After possibly swapping the order of our disjoint cycles, and cyclically permuting the elements of the cycle that contains i and j, we may assume that our disjoint cycle decomposition of π is given by

$$\pi \;\; = \;\; (i \;\, a_1 \dots a_p \;\, j \;\, b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r).$$

Claim. *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

Proof of the Claim (continued). Reminder: $\tau = (ij)$.

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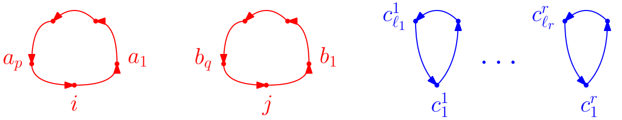
In the permutation $\tau \circ \pi$, the red cycle essentially gets "split up" into two, while the blue cycles remain unaffected, as follows (next slide):

$$\begin{aligned}\tau \circ \pi &= (ij) \circ (i\ a_1 \dots a_p\ j\ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r) \\ &= \underbrace{(i\ a_1 \dots a_p)(j\ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r)}_{=:\pi'}.\end{aligned}$$



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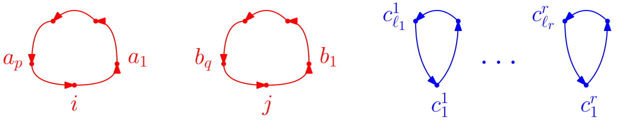
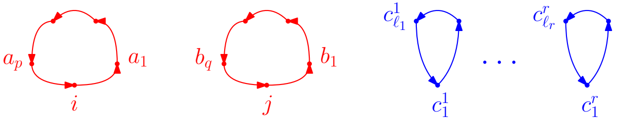


We now see that the disjoint cycle decomposition of $\tau \circ \pi$ has one cycle more than the disjoint cycle decomposition of π,

$$\begin{aligned}\tau \circ \pi &= (ij) \circ (i\ a_1 \dots a_p\ j\ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r) \\ &= \underbrace{(i\ a_1 \dots a_p)(j\ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r)}_{=:\pi'}.\end{aligned}$$



We now see that the disjoint cycle decomposition of $\tau \circ \pi$ has one cycle more than the disjoint cycle decomposition of π, and it follows that $\mathrm{sgn}(\tau \circ \pi) = -\mathrm{sgn}(\pi)$.

- Indeed, the disjoint cycle decomposition of π has $r+1$ cycles, whereas the disjoint cycle decomposition of $\tau \circ \pi$ has $r+2$ cycles. Therefore, $\mathrm{sgn}(\tau \circ \pi) = (-1)^{n-(r+2)} = (-1)^{n-(r+1)-1} = -(-1)^{n-(r+1)} = -\mathrm{sgn}(\pi)$.

Claim. *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

Proof of the Claim (continued). Reminder: $\tau = (ij)$.

Case 2: i and j are in different cycles of the disjoint cycle decomposition of π.

Claim. *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

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Proof of the Claim (continued). Reminder: $\tau = (ij)$.

Case 2: i and j are in different cycles of the disjoint cycle decomposition of π. After possibly swapping the order of our disjoint cycles, and cyclically permuting the elements of the cycles that contain i and j, we may assume that our disjoint cycle decomposition of π is given by

$$\pi = (i\ a_1 \dots a_p)(j\ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r)$$

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Proof of the Claim (continued). Reminder: $\tau = (ij)$.

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$$\pi = (i\ a_1 \dots a_p)(j\ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r)$$

We then have that

$$\begin{aligned} \pi &= (i\ a_1 \dots a_p)(j\ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r) \\ &\overset{(*)}{=} (ij) \circ (i\ a_1 \dots a_p\ j\ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r), \end{aligned}$$

where (*) follows from the argument given in Case 1.

Claim. *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

Proof of the Claim (continued). Reminder: $\tau = (ij)$.

Case 2: i and j are in different cycles of the disjoint cycle decomposition of π. After possibly swapping the order of our disjoint cycles, and cyclically permuting the elements of the cycles that contain i and j, we may assume that our disjoint cycle decomposition of π is given by

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where (*) follows from the argument given in Case 1. We now compose both sides with $\tau = (ij)$ on the left, and we obtain (next slide):

$$(ij) \circ \pi \quad = \quad (ij) \circ (ij) \circ (i \ a_1 \dots a_p \ j \ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r).$$

$$(ij) \circ \pi \quad = \quad (ij) \circ (ij) \circ (i \; a_1 \ldots a_p \; j \; b_1 \ldots b_q)(c_1^1 \ldots c_{\ell_1}^1) \ldots (c_1^r \ldots c_{\ell_r}^r).$$

Since $(ij) = \tau$ and $(ij) \circ (ij) = 1_n$, we deduce that

$$\tau \circ \pi \quad = \quad (i \; a_1 \ldots a_p \; j \; b_1 \ldots b_q)(c_1^1 \ldots c_{\ell_1}^1) \ldots (c_1^r \ldots c_{\ell_r}^r).$$

$$(ij) \circ \pi \quad = \quad (ij) \circ (ij) \circ (i \; a_1 \dots a_p \; j \; b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r).$$

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$$\tau \circ \pi \quad = \quad (i \; a_1 \dots a_p \; j \; b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r).$$

As we can see, in the permutation $\tau \circ \pi$, the two red cycles of π essentially get "merged" into one, while the blue cycles remain unaffected.

$$(ij) \circ \pi \quad = \quad (ij) \circ (ij) \circ (i \ a_1 \dots a_p \ j \ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r).$$

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$$\tau \circ \pi \quad = \quad (i \ a_1 \dots a_p \ j \ b_1 \dots b_q)(c_1^1 \dots c_{\ell_1}^1) \dots (c_1^r \dots c_{\ell_r}^r).$$

As we can see, in the permutation $\tau \circ \pi$, the two red cycles of π essentially get "merged" into one, while the blue cycles remain unaffected. But now the disjoint cycle decomposition of $\tau \circ \pi$ has one cycle less than the disjoint cycle decomposition of π, and it follows that $\text{sgn}(\tau \circ \pi) = -\text{sgn}(\pi)$.

- Indeed, the disjoint cycle decomposition of π has $r + 2$ cycles, whereas the disjoint cycle decomposition of $\tau \circ \pi$ has $r + 1$ cycles. Therefore, $\text{sgn}(\tau \circ \pi) = (-1)^{n-(r+1)} = (-1)^{n-(r+2)+1} = -(-1)^{n-(r+2)} = -\text{sgn}(\pi)$.

This completes the proof of the Claim. ♦

Proposition 2.3.5

Let $n \geq 2$ be an integer. Then for all $\pi, \tau \in S_n$ s.t. τ is a **transposition**, we have that $\text{sgn}(\tau \circ \pi) = \text{sgn}(\pi \circ \tau) = -\text{sgn}(\pi)$.

Proof (continued). We have now proven the Claim below.

> ***Claim.*** *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

Proposition 2.3.5

Let $n \geq 2$ be an integer. Then for all $\pi, \tau \in S_n$ s.t. τ is a **transposition**, we have that $\text{sgn}(\tau \circ \pi) = \text{sgn}(\pi \circ \tau) = -\text{sgn}(\pi)$.

Proof (continued). We have now proven the Claim below.

> ***Claim.*** *For all $\pi, \tau \in S_n$ s.t. τ is a transposition, we have that $sgn(\tau \circ \pi) = -sgn(\pi)$.*

Now, fix $\pi, \tau \in S_n$ s.t. τ is a transposition. By the Claim, we have that $\text{sgn}(\tau \circ \pi) = -\pi$. On the other hand,

$$\begin{array}{rcll} \text{sgn}(\pi \circ \tau) & = & \text{sgn}\Big((\pi \circ \tau)^{-1}\Big) & \text{by Proposition 2.3.2} \\ & = & \text{sgn}\big(\tau^{-1} \circ \pi^{-1}\big) & \text{by Proposition 1.10.17(c) (or by Proposition 2.2.4(f))} \\ & = & \text{sgn}\big(\tau \circ \pi^{-1}\big) & \text{because } \tau \text{ is a transposition, and so } \tau^{-1} = \tau \\ & = & -\text{sgn}(\pi^{-1}) & \text{by the Claim applied to } \pi^{-1} \text{ and } \tau \\ & = & -\text{sgn}(\pi) & \text{by Proposition 2.3.2.} \end{array}$$

$\square$

Proposition 2.3.5

Let $n \geq 2$ be an integer. Then for all $\pi, \tau \in S_n$ s.t. τ is a **transposition**, we have that $\text{sgn}(\tau \circ \pi) = \text{sgn}(\pi \circ \tau) = -\text{sgn}(\pi)$.

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Let $n \geq 2$ be an integer. Then for all $\pi, \tau \in S_n$ s.t. τ is a **transposition**, we have that $\text{sgn}(\tau \circ \pi) = \text{sgn}(\pi \circ \tau) = -\text{sgn}(\pi)$.

Theorem 2.3.6

Let $n \geq 2$. Then for any permutation $\pi \in S_n$, if π can be expressed as a composition of r transpositions, then

(a) $\text{sgn}(\pi) = (-1)^r$;

(b) π is an even permutation iff r is even;

(c) π is an odd permutation iff r is odd.

Theorem 2.3.7

Let $n \geq 2$ be an integer, and let $\sigma, \pi \in S_n$. Then $\mathrm{sgn}(\sigma \circ \pi) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\pi)$.

Proof.

Theorem 2.3.7

Let $n \geq 2$ be an integer, and let $\sigma, \pi \in S_n$. Then $\mathrm{sgn}(\sigma \circ \pi) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\pi)$.

Proof. This easily follows from Proposition 2.3.3 and Theorem 2.3.6.

Theorem 2.3.7

Let $n \geq 2$ be an integer, and let $\sigma, \pi \in S_n$. Then $\operatorname{sgn}(\sigma \circ \pi) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)$.

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- $\sigma = (s_1 s_1') \circ (s_2 s_2') \circ \cdots \circ (s_k s_k')$;
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So, $\text{sgn}(\sigma \circ \pi) = (-1)^{k+\ell} = (-1)^k(-1)^\ell = \text{sgn}(\sigma)\text{sgn}(\pi)$. $\square$

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- We apply Theorem 2.2.9.

Theorem 2.2.9

Let $(G, \circ)$ be a group with identity element e, and with the inverse of an element $a \in G$ denoted by a^{-1}. Then for all $H \subseteq G$, we have that $(H, \circ)$ is a subgroup of $(G, \circ)$ iff all the following hold:

(i) $e \in H$;

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- Theorem 2.2.9 now guarantees that A_n is indeed a subgroup of S_n.

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- We remark that the set of odd permutations in S_n ($n \geq 2$), call it O_n, does **not** form a subgroup of S_n.
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- However, A_n is indeed the standard notation for the set of even permutations in S_n.

Definition

Let n be a positive integer. An *inversion* of a permutation $\pi \in S_n$ is an ordered pair (i, j) of numbers in $\{1, \ldots, n\}$ s.t. $i < j$ and $\pi(i) > \pi(j)$.

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The permutation

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Induction step: Fix a non-negative integer r, and assume inductively that any permutation in S_n that has exactly r inversions has sign $(-1)^r$. WTS any permutation in S_n that has exactly $r + 1$ inversions has sign $(-1)^{r+1}$.

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Now, consider the transposition $\tau := \big(\pi(p)\pi(p+1)\big)$ in S_n, and set $\pi' := \tau \circ \pi$, so that

$$\pi' = \begin{pmatrix} 1 & \dots & p-1 & p & p+1 & p+2 & \dots & n \\ \pi(1) & \dots & \pi(p-1) & \pi(p+1) & \pi(p) & \pi(p+2) & \dots & \pi(n) \end{pmatrix}.$$

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But now

$$
\begin{array}{rcll}
(-1)^r & = & \mathrm{sgn}(\pi') & \text{by the induction hypothesis,} \\
 & & & \text{since } \pi' \text{ has exactly } r \text{ inversions} \\
 & = & \mathrm{sgn}(\tau \circ \pi) & \text{because } \pi' = \tau \circ \pi \\
 & = & -\mathrm{sgn}(\pi) & \text{by Proposition 2.3.5,} \\
 & & & \text{since } \tau \text{ is a transposition,}
\end{array}
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and it follows that $\mathrm{sgn}(\pi) = (-1)^{r+1}$. This completes the induction. $\square$

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- Note that π' has three inversions, whereas π has four.

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$$\pi = \begin{pmatrix} 1 & 2 & 3 & {\color{red}4} & {\color{blue}5} & 6 & 7 \\ 2 & 1 & 3 & {\color{red}6} & {\color{blue}5} & 4 & 7 \end{pmatrix}.$$

- If we had, instead, chosen an arbitrary inversion of π, then the number of inversions would not necessarily decrease by one, and we could not apply the induction hypothesis.

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- Note that π'' has only one inversion (namely, $(1, 2)$), whereas π has four.

Definition

A *field* is an ordered triple $(\mathbb{F}, +, \cdot)$, where $\mathbb{F}$ is a set, and $+$ and $\cdot$ are binary operations on $\mathbb{F}$ (i.e. functions from $\mathbb{F} \times \mathbb{F}$ to $\mathbb{F}$), called *addition* and *multiplication*, respectively, satisfying the following axioms:

1. addition and multiplication are associative, that is, for all $a, b, c \in \mathbb{F}$, we have that $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
2. addition and multiplication are commutative, that is, for all $a, b \in \mathbb{F}$, we have that $a + b = b + a$ and $a \cdot b = b \cdot a$;
3. there exist distinct elements $0_{\mathbb{F}}, 1_{\mathbb{F}} \in \mathbb{F}$ s.t. for all $a \in \mathbb{F}$, $a + 0_{\mathbb{F}} = a$ and $a \cdot 1_{\mathbb{F}} = a$; $0_{\mathbb{F}}$ is called the *additive identity* of $\mathbb{F}$, and $1_{\mathbb{F}}$ is called the *multiplicative identity* of $\mathbb{F}$;
4. for every $a \in \mathbb{F}$, there exists an element in $\mathbb{F}$, denoted by $-a$ and called the *additive inverse* of a, s.t. $a + (-a) = 0_{\mathbb{F}}$;
5. for all $a \in \mathbb{F} \setminus \{0_{\mathbb{F}}\}$, there exists an element in $\mathbb{F}$, denoted by a^{-1} and called the *multiplicative inverse* of a, s.t. $a \cdot a^{-1} = 1_{\mathbb{F}}$;
6. multiplication is distributive over addition, that is, for all $a, b, c \in \mathbb{F}$, we have that $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Example 2.4.1

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- **Remark:** Axioms 1, 2, and 3 imply that $(\mathbb{F}, +)$ and $(\mathbb{F}, \cdot)$ are monoids with identity elements $0_{\mathbb{F}}$ and $1_{\mathbb{F}}$, respectively. Proposition 2.1.1 guarantees that $0_{\mathbb{F}}$ and $1_{\mathbb{F}}$ are unique.
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     - Indeed, for $a, b, c \in \mathbb{F}$, we have that $(b + c)a \overset{\text{ax. 2.}}{=} a(b + c) \overset{\text{ax. 6.}}{=} ab + ac \overset{\text{ax. 2.}}{=} ba + ca$.

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Proof (continued). Reminder: $a0 = a0 + a0$.

0	$=$	$-(a0) + a0$	because $-(a0)$ is the additive inverse of $a0$
	$=$	$-(a0) + (a0 + a0)$	because $a0 = a0 + a0$ (proven above)
	$=$	$(-(a0) + a0) + a0$	because $+$ is associative
	$=$	$0 + a0$	because $-(a0)$ is the additive inverse of $a0$
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Thus, $a0 = 0$. This proves (a).

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$$a = a \cdot 1 = a(bb^{-1}) \overset{(*)}{=} (ab)b^{-1} \overset{(**)}{=} 0b^{-1} \overset{(***)}{=} 0,$$

where (*) follows from the associativity of multiplication, (**) follows from the fact that $ab = 0$, and (***) follows from (a).

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$$0 \stackrel{(*)}{=} 0a = (1-1)a = 1a + (-1)a = a + (-1)a,$$

where (*) follows from (a). Consequently,

$$\begin{aligned}
-a &= -a + 0 && \text{because 0 is the additive identity of the field } \mathbb{F} \\
&= -a + (a + (-1)a) && \text{because } 0 = a + (-1)a \text{ (proven above)} \\
&= (-a + a) + (-1)a && \text{because } + \text{ is associative} \\
&= 0 + (-1)a && \text{because } -a \text{ is the additive inverse of } a \\
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This proves (c). $\square$

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(b) for all $a, b \in \mathbb{F}$, if $ab = 0$, then $a = 0$ or $b = 0$;

(c) for all $a \in \mathbb{F}$, $(-1)a = -a$.

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Further, $0 := [0]_p$ is the additive identity and $1 := [1]_p$ is the multiplicative identity of $(\mathbb{Z}_p, +, \cdot)$. Moreover, $[0]_p \neq [1]_p$, since $0 \not\equiv 1 \pmod{p}$. Thus, $(\mathbb{Z}_p, +, \cdot)$ satisfies axiom 3 from the definition of a field.

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Further, for all $a \in \mathbb{Z}$, the additive inverse of $[a]_p$ in $(\mathbb{Z}_p, +, \cdot)$ is $[-a]_p$, and so axiom 4 is satisfied.

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Proof (continued). Finally, by Fermat's Little Theorem, every number $a \in \mathbb{Z}_p \setminus \{0\}$ has a multiplicative inverse, namely, a^{p-2}, and it follows that axiom 5 is satisfied.

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This proves that $(\mathbb{Z}_p, +, \cdot)$ is indeed a field, which is what we needed to show. $\square$

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- **Remark:** For a positive integer n that is **not** prime, $(\mathbb{Z}_n, +, \cdot)$ is not a field.
    - If $n = 1$, then this follows from the fact that $\mathbb{Z}_n = \mathbb{Z}_1$ has only one element, whereas every field has at least two elements (namely, the additive and multiplicative identities, which cannot be equal by axiom 3 of the definition of a field).

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    - Now, let us suppose that $n \geq 2$ is composite, say $n = pq$ where $p, q \geq 2$ are integers. Then $[p]_n[q]_n = [pq]_n = [n]_n = 0$. So, if $(\mathbb{Z}_n, +, \cdot)$ were a field, Proposition 2.4.2(b) would imply that at least one of $[p]_n$ and $[q]_n$ is 0, a contradiction.

Theorem 2.4.4

Let $n \geq 2$ be an integer. Then there exists a field of size n iff n is a power of a prime.[a] Moreover, if n is a power of a prime, then up to "isomorphism" (i.e. up to renaming the operations and elements of the field), there is exactly one field of size n, and it is denoted by $\mathbb{F}_n$.[b]

[a] "n is a power of a prime" means that there exists some prime number p and a positive integer m s.t. $n = p^m$.

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- **Remark:** For a prime number p, we have that $\mathbb{F}_p = \mathbb{Z}_p$.
  - However, if $n = p^m$, where p is a prime number and $m \geq 2$ is an integer, then $\mathbb{F}_n \neq \mathbb{Z}_n$ (this is because $\mathbb{F}_n$ is a field, but $\mathbb{Z}_n$ is **not** a field).

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- It is sometimes more convenient to use the notation $\frac{1}{a}$ instead of a^{-1}, and $\frac{a}{b}$ instead of $b^{-1}a$.
- However, when working over a finite field such as $\mathbb{Z}_p$ (for a prime number p), we **never** leave a fraction as a final answer, and instead, we always simplify.

Definition

The *characteristic* of a field $\mathbb{F}$ is the smallest positive integer n (if it exists) s.t. in the field $\mathbb{F}$, we have that

$$\underbrace{1 + \cdots + 1}_{n} \;=\; 0,$$

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Since $\mathbb{F}$ is a field, Proposition 2.4.2(b) guarantees that at least one of the numbers $\underbrace{1+\dots+1}_{p}$ and $\underbrace{1+\dots+1}_{q}$ is zero. But this is impossible since $0 < p, q < \text{char}(\mathbb{F})$. $\square$