Linear Algebra 1

Lecture #3

Matrix multiplication and transpose. Solving matrix equations of the form $AX = B$ and $XA = B$

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  - $A + B := \begin{bmatrix} a_{i,j} + b_{i,j} \end{bmatrix}_{n\times m}$;
  - $A - B := \begin{bmatrix} a_{i,j} - b_{i,j} \end{bmatrix}_{n\times m}$;
  - $cA := \begin{bmatrix} ca_{i,j} \end{bmatrix}$.

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  - $cA := \left[\begin{array}{c} ca_{i,j} \end{array}\right]$.
- Thus, we add (resp. subtract) matrices by adding (resp. subtracting) corresponding entries, i.e.
  - $\left[\begin{array}{c} a_{i,j} \end{array}\right]_{n\times m} + \left[\begin{array}{c} b_{i,j} \end{array}\right]_{n\times m} = \left[\begin{array}{c} a_{i,j} + b_{i,j} \end{array}\right]_{n\times m}$;
  - $\left[\begin{array}{c} a_{i,j} \end{array}\right]_{n\times m} - \left[\begin{array}{c} b_{i,j} \end{array}\right]_{n\times m} = \left[\begin{array}{c} a_{i,j} - b_{i,j} \end{array}\right]_{n\times m}$.

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- Thus, we add (resp. subtract) matrices by adding (resp. subtracting) corresponding entries, i.e.
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- Similarly, we multiply a matrix by a scalar (on the left) by multiplying each entry of the matrix by that scalar, i.e.
  - $c \left[\begin{array}{c} a_{i,j} \end{array} \right]_{n\times m} = \left[\begin{array}{c} ca_{i,j} \end{array} \right]_{n\times m}$.

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- We define

$$AB \;\; := \;\; \left[\begin{array}{ccc} A\mathbf{b}_1 & \dots & A\mathbf{b}_p \end{array} \right]$$

Note that $AB \in \mathbb{F}^{n\times p}$.

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Note that $AB \in \mathbb{F}^{n\times p}$.
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- The matrix AB has the same number of **rows** as A, and the same number of **columns** as B.

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- The matrix AB has the same number of **rows** as A, and the same number of **columns** as B.
- Schematically, we get:
$$(n \times m) \;\; \cdot \;\; (m \times p) \;\; = \;\; (n \times p).$$

Example 1.7.1

Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 0 & -1 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute AB.

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$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

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so that $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$. Then $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix}$. We compute $A\mathbf{b}_1 = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$ and $A\mathbf{b}_2 = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$, which yields $AB = \begin{bmatrix} -3 & 4 \\ 6 & -4 \end{bmatrix}$. $\square$

Proposition 1.7.2

Let $\mathbb{F}$ be a field, let m, n, p be positive integers, and let $A \in \mathbb{F}^{n \times m}$ be a matrix. Then all the following hold:

(a) $I_n A = A I_m = A$;

(b) $A O_{m \times p} = O_{n \times p}$;

(c) $O_{p \times n} A = O_{p \times m}$.

Proof.

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Proof. Parts (b) and (c) readily follow from the appropriate definitions (the details are left as an easy exercise). Let us prove (a).

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Proof. Parts (b) and (c) readily follow from the appropriate definitions (the details are left as an easy exercise). Let us prove (a). Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$. To show that $I_n A = A$, we compute (next slide):

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Proof (continued). Reminder: $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$.

$$\begin{array}{rcll} I_n A & = & I_n \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right] & \\ & = & \left[\begin{array}{ccc} I_n\mathbf{a}_1 & \dots & I_n\mathbf{a}_m \end{array} \right] & \text{by the definition of matrix multiplication} \\ & = & \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right] & \text{by Proposition 1.4.5} \\ & = & A. & \end{array}$$

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On the other hand, to show that $AI_m = A$, we compute:

$$\begin{aligned} AI_m &= A\begin{bmatrix} \mathbf{e}_1^m & \dots & \mathbf{e}_m^m \end{bmatrix} \\ &= \begin{bmatrix} A\mathbf{e}_1^m & \dots & A\mathbf{e}_m^m \end{bmatrix} && \text{by the definition of matrix multiplication} \\ &= \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix} && \text{by Proposition 1.4.4} \\ &= A. \end{aligned}$$

This proves (a). $\square$

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(a) $I_n A = A I_m = A$;

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- First, suppose we are given a matrix $A \in \mathbb{F}^{n \times m}$ and a vector $\mathbf{v} \in \mathbb{F}^m$.
- Set $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times m}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$.
- Then by the definition of a matrix-vector product, we have that $A\mathbf{v} \in \mathbb{F}^n$, and moreover, the i-th entry of the vector $A\mathbf{v}$ is

$$\sum_{k=1}^{m} a_{i,k} v_k \quad = \quad a_{i,1} v_1 + \cdots + a_{i,k} v_k + \cdots + a_{i,m} v_m.$$

  - Justification: next slide.

$$
\begin{aligned}
A\mathbf{v} &= \begin{bmatrix} a_{1,1} & \dots & a_{1,k} & \dots & a_{1,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \dots & a_{i,k} & \dots & a_{i,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,k} & \dots & a_{n,m} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_k \\ \vdots \\ v_m \end{bmatrix} \\
&= v_1 \begin{bmatrix} a_{1,1} \\ \vdots \\ a_{i,1} \\ \vdots \\ a_{n,1} \end{bmatrix} + \cdots + v_k \begin{bmatrix} a_{1,k} \\ \vdots \\ a_{i,k} \\ \vdots \\ a_{n,k} \end{bmatrix} + \cdots + v_m \begin{bmatrix} a_{1,m} \\ \vdots \\ a_{i,m} \\ \vdots \\ a_{n,m} \end{bmatrix} \\
&= \begin{bmatrix} a_{1,1}v_1 + \cdots + a_{1,k}v_k + \cdots + a_{1,m}v_m \\ \vdots \\ a_{i,1}v_1 + \cdots + a_{i,k}v_k + \cdots + a_{i,m}v_m \\ \vdots \\ a_{n,1}v_1 + \cdots + a_{n,k}v_k + \cdots + a_{n,m}v_m \end{bmatrix}.
\end{aligned}
$$

- Suppose now that we are given matrices $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times p}$, and set $A = \left[\begin{array}{c} a_{i,j} \end{array}\right]_{n\times m}$ and $B = \left[\begin{array}{c} b_{i,j} \end{array}\right]_{m\times p}$.

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- The matrix AB belongs to $\mathbb{F}^{n \times p}$.
- We would like to compute the i, j-th entry of the matrix AB in terms of the entries of A and B.
- The i, j-th entry of AB is precisely the i-th entry of the j-th column of AB, and by the definition of matrix product, the j-th column of AB is the vector $A\mathbf{b}_j$, where $\mathbf{b}_j = \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{m,j} \end{bmatrix}$ is the j-th column of B.

- Suppose now that we are given matrices $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times p}$, and set $A = \left[\ a_{i,j}\ \right]_{n\times m}$ and $B = \left[\ b_{i,j}\ \right]_{m\times p}$.
- The matrix AB belongs to $\mathbb{F}^{n\times p}$.
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- The i, j-th entry of AB is precisely the i-th entry of the j-th column of AB, and by the definition of matrix product, the j-th column of AB is the vector $A\mathbf{b}_j$, where $\mathbf{b}_j = \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{m,j} \end{bmatrix}$ is the j-th column of B.
- Using the formula for the matrix-vector product that we obtained above, we see that the i-th entry of the vector $A\mathbf{b}_j$ is $\sum\limits_{k=1}^{m} a_{i,k} b_{k,j}$.

- Suppose now that we are given matrices $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$, and set $A = \left[\begin{array}{c} a_{i,j} \end{array}\right]_{n \times m}$ and $B = \left[\begin{array}{c} b_{i,j} \end{array}\right]_{m \times p}$.
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- Using the formula for the matrix-vector product that we obtained above, we see that the i-th entry of the vector $A\mathbf{b}_j$ is $\sum\limits_{k=1}^{m} a_{i,k} b_{k,j}$.
- So, the i, j-th entry of the $n \times p$ matrix AB is $\sum\limits_{k=1}^{m} a_{i,k} b_{k,j}$.

- Another way to write this is as follows:

$$\left[\ a_{i,j}\ \right]_{n\times m}\ \left[\ b_{i,j}\ \right]_{m\times p}\ =\ \left[\ \sum_{k=1}^{m} a_{i,k}b_{k,j}\ \right]_{n\times p},$$

  where in each of the three matrices, the expression between the square brackets is the general form of the i,j-th entry (i.e. the entry in the i-th row and j-th column) of the matrix in question.

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  where in each of the three matrices, the expression between the square brackets is the general form of the i,j-th entry (i.e. the entry in the i-th row and j-th column) of the matrix in question.
- To obtain the i,j-th entry of the matrix AB, we focus on the i-th row of A and j-th column of B.
- We then take the sum of the products of the corresponding entries of this row and column, and we obtain the i,j-th entry of AB.
  - Diagram: next slide.

$$
\begin{array}{cc}
 & \begin{bmatrix} b_{1,1} & \dots & b_{1,j} & \dots & b_{1,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k,1} & \dots & b_{k,j} & \dots & b_{k,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{m,1} & \dots & b_{m,j} & \dots & b_{m,p} \end{bmatrix} \\
\begin{bmatrix} a_{1,1} & \dots & a_{1,k} & \dots & a_{1,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \dots & a_{i,k} & \dots & a_{i,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,k} & \dots & a_{n,m} \end{bmatrix} & \begin{bmatrix} \\ \\ \displaystyle\sum_{k=1}^{m} a_{i,k} b_{k,j} \\ \\ \\ \end{bmatrix}
\end{array}
$$

Example 1.7.3

Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_2$. Compute the matrix AB.

Solution.

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Solution. We compute as shown below (the rows of A are color coded, as are the columns of B).

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1\cdot 1 + 0 \cdot 1 & 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \\ 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 0 + 1 \cdot 1 & 1 \cdot 1 + 1 \cdot 0 \end{bmatrix} \end{array}$$

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with entries understood to be in $\mathbb{Z}_2$. Compute the matrix AB.

Solution (continued).

$$AB = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 1 & 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \\ 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 0 + 1 \cdot 1 & 1 \cdot 1 + 1 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

□

Theorem 1.7.5

For any matrices A, B, and C, and any scalars α and β, the following hold (provided the matrices are of compatible size for the operation in question, and the entries of our matrices and our scalars all belong to the same field $\mathbb{F}$):

a. $(\alpha + \beta)A = \alpha A + \beta A$;
b. $(\alpha\beta)A = \alpha(\beta A)$
c. $A + B = B + A$;
d. $(A + B) + C = A + (B + C)$;
e. $(A + B)C = AC + BC$;
f. $A(B + C) = AB + AC$;
g. $(AB)C = A(BC)$;
h. $(\alpha A)B = \alpha(AB)$;
i. $A(\alpha B) = \alpha(AB)$.

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Proof of (g). Fix matrices $A = \left[\ a_{i,j}\ \right]_{n_1 \times n_2}$ in $\mathbb{F}^{n_1 \times n_2}$, $B = \left[\ b_{i,j}\ \right]_{n_2 \times n_3}$ in $\mathbb{F}^{n_2 \times n_3}$, and $C = \left[\ c_{i,j}\ \right]_{n_3 \times n_4}$ in $\mathbb{F}^{n_3 \times n_4}$.

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Clearly, both $(AB)C$ and $A(BC)$ are matrices in $\mathbb{F}^{n_1 \times n_4}$. To prove that these two matrices are equal, it suffices to prove that their corresponding entries are equal. So, fix indices $i \in \{1, \dots, n_1\}$ and $j \in \{1, \dots, n_4\}$. We must show that the i,j-th entry of $(AB)C$ is equal to the i,j-th entry of $A(BC)$.

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Proof of (g) (continued). We first compute the i,j-th entry of $(AB)C$. The i-th row of the $n_1 \times n_3$ matrix AB is

$$\left[\begin{array}{cccc} \sum_{k=1}^{n_2} a_{i,k} b_{k,1} & \sum_{k=1}^{n_2} a_{i,k} b_{k,2} & \dots & \sum_{k=1}^{n_2} a_{i,k} b_{k,n_3} \end{array}\right].$$

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Proof of (g) (continued). We first compute the i, j-th entry of $(AB)C$. The i-th row of the $n_1 \times n_3$ matrix AB is $\begin{bmatrix} \sum_{k=1}^{n_2} a_{i,k}b_{k,1} & \sum_{k=1}^{n_2} a_{i,k}b_{k,2} & \dots & \sum_{k=1}^{n_2} a_{i,k}b_{k,n_3} \end{bmatrix}$. The j-th column of the $n_3 \times n_4$ matrix C is $\begin{bmatrix} c_{1,j} \\ c_{2,j} \\ \vdots \\ c_{n_3,j} \end{bmatrix}$.

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So, the i, j-th entry of the $n_1 \times n_4$ matrix $(AB)C$ is

$$\sum_{\ell=1}^{n_3} \Big((\sum_{k=1}^{n_2} a_{i,k}b_{k,\ell})c_{\ell,j} \Big).$$

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(g) $(AB)C = A(BC)$

Proof of (g) (continued). We now compute the i, j-th entry of $A(BC)$. The i-th row of the $n_1 \times n_2$ matrix A is $\begin{bmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n_2} \end{bmatrix}$. The j-th column of the $n_2 \times n_4$ matrix

$$BC \text{ is } \begin{bmatrix} \sum_{k=1}^{n_3} b_{1,k} c_{k,j} \\ \sum_{k=1}^{n_3} b_{2,k} c_{k,j} \\ \vdots \\ \sum_{k=1}^{n_3} b_{n_2,k} c_{k,j} \end{bmatrix}.$$

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Proof of (g) (continued). We now compute the i, j-th entry of $A(BC)$. The i-th row of the $n_1 \times n_2$ matrix A is $\begin{bmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n_2} \end{bmatrix}$. The j-th column of the $n_2 \times n_4$ matrix

$$BC \text{ is } \begin{bmatrix} \sum_{k=1}^{n_3} b_{1,k} c_{k,j} \\ \sum_{k=1}^{n_3} b_{2,k} c_{k,j} \\ \vdots \\ \sum_{k=1}^{n_3} b_{n_2,k} c_{k,j} \end{bmatrix}.$$

So, the i, j-th entry of the $n_1 \times n_4$ matrix $(AB)C$ is

$$\sum_{\ell=1}^{n_2} \left(a_{i,\ell} \Big(\sum_{k=1}^{n_3} b_{\ell,k} c_{k,j} \Big) \right).$$

(g) $(AB)C = A(BC)$

Proof of (g) (continued). Reminder:

- the i, j-th entry of $(AB)C$ is $\sum_{\ell=1}^{n_3} \left((\sum_{k=1}^{n_2} a_{i,k} b_{k,\ell}) c_{\ell,j} \right)$;
- the i, j-th entry of $A(BC)$ is $\sum_{\ell=1}^{n_2} \left(a_{i,\ell} (\sum_{k=1}^{n_3} b_{\ell,k} c_{k,j}) \right)$.

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Proof of (g) (continued). Reminder:

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It now remains to show that

$$\sum_{\ell=1}^{n_3} \left((\sum_{k=1}^{n_2} a_{i,k} b_{k,\ell}) c_{\ell,j} \right) \quad = \quad \sum_{\ell=1}^{n_2} \left(a_{i,\ell} (\sum_{k=1}^{n_3} b_{\ell,k} c_{k,j}) \right).$$

(g) $(AB)C = A(BC)$

Proof of (g) (continued). Reminder:

- the i,j-th entry of $(AB)C$ is $\sum\limits_{\ell=1}^{n_3}\Big((\sum\limits_{k=1}^{n_2} a_{i,k}b_{k,\ell})c_{\ell,j}\Big)$;
- the i,j-th entry of $A(BC)$ is $\sum\limits_{\ell=1}^{n_2}\Big(a_{i,\ell}(\sum\limits_{k=1}^{n_3} b_{\ell,k}c_{k,j})\Big)$.

It now remains to show that

$$\sum_{\ell=1}^{n_3}\Big((\sum_{k=1}^{n_2} a_{i,k}b_{k,\ell})c_{\ell,j}\Big) \quad = \quad \sum_{\ell=1}^{n_2}\Big(a_{i,\ell}(\sum_{k=1}^{n_3} b_{\ell,k}c_{k,j})\Big).$$

For this, we compute (next slide):

(g) $(AB)C = A(BC)$

Proof of (g) (continued).

$$\begin{aligned}
\sum_{\ell=1}^{n_3}\Big(\big(\sum_{k=1}^{n_2} a_{i,k}b_{k,\ell}\big)c_{\ell,j}\Big) &= \sum_{\ell=1}^{n_3}\Big(\sum_{k=1}^{n_2} a_{i,k}b_{k,\ell}c_{\ell,j}\Big) \\
&= \sum_{k=1}^{n_2}\Big(\sum_{\ell=1}^{n_3} a_{i,k}b_{k,\ell}c_{\ell,j}\Big) \\
&= \sum_{k=1}^{n_2}\Big(a_{i,k}\big(\sum_{\ell=1}^{n_3} b_{k,\ell}c_{\ell,j}\big)\Big) \\
&= \sum_{\ell=1}^{n_2}\Big(a_{i,\ell}\big(\sum_{k=1}^{n_3} b_{\ell,k}c_{k,j}\big)\Big)
\end{aligned}$$

and we obtain the equality that we needed. This proves (g). □

Theorem 1.7.5

For any matrices A, B, and C, and any scalars α and β, the following hold (provided the matrices are of compatible size for the operation in question, and the entries of our matrices and our scalars all belong to the same field $\mathbb{F}$):

a. $(\alpha + \beta)A = \alpha A + \beta A$;
b. $(\alpha\beta)A = \alpha(\beta A)$
c. $A + B = B + A$;
d. $(A + B) + C = A + (B + C)$;
e. $(A + B)C = AC + BC$;
f. $A(B + C) = AB + AC$;
g. $(AB)C = A(BC)$;
h. $(\alpha A)B = \alpha(AB)$;
i. $A(\alpha B) = \alpha(AB)$.

- **Warning:** Matrix multiplication is **not** commutative, that is, for matrices A and B,

$$AB \neq BA.$$

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    - For instance, if $A \in \mathbb{F}^{2\times 3}$ and $B \in \mathbb{F}^{3\times 4}$, where $\mathbb{F}$ is some field, then AB is defined, but BA is not.

- **Warning:** Matrix multiplication is **not** commutative, that is, for matrices A and B,

$$AB \not\asymp BA.$$

- In fact, it is possible that one of AB and BA is defined, while the other one is not.
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- Moreover, it is possible that both AB and BA are defined, but are not of the same size.
  - For instance, if $A \in \mathbb{F}^{2\times 3}$ and $B \in \mathbb{F}^{3\times 2}$, where $\mathbb{F}$ is some field, then $AB \in \mathbb{F}^{2\times 2}$ and $BA \in \mathbb{F}^{3\times 3}$.

- **Warning:** Matrix multiplication is **not** commutative, that is, for matrices A and B,

$$AB \cancel{=} BA.$$

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  - For instance, if $A \in \mathbb{F}^{2\times 3}$ and $B \in \mathbb{F}^{3\times 4}$, where $\mathbb{F}$ is some field, then AB is defined, but BA is not.
- Moreover, it is possible that both AB and BA are defined, but are not of the same size.
  - For instance, if $A \in \mathbb{F}^{2\times 3}$ and $B \in \mathbb{F}^{3\times 2}$, where $\mathbb{F}$ is some field, then $AB \in \mathbb{F}^{2\times 2}$ and $BA \in \mathbb{F}^{3\times 3}$.
- Finally, it is possible that AB and BA are both defined, and are of the same size, but $AB \neq BA$.
  - For example, for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we have that $AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and so $AB \neq BA$.

Corollary 1.7.6

For any matrices A, B, vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and scalars α and β, the following hold (provided the matrices and vectors are of compatible size for the operation in question, and the entries of our matrices, the entries of our vectors, and our scalars all belong to the same field $\mathbb{F}$):

(a) $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$;

(b) $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$

(c) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;

(d) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;

(e) $(A + B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}$;

(f) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

(g) $(AB)\mathbf{u} = A(B\mathbf{u})$;

(h) $(\alpha A)\mathbf{u} = \alpha(A\mathbf{u})$;

(i) $A(\alpha\mathbf{u}) = \alpha(A\mathbf{u})$.

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- For a field $\mathbb{F}$ and a square matrix $A \in \mathbb{F}^{n \times n}$, we define
  - $A^0 := I_n$;
  - $A^{m+1} := A^m A$ for all non-negative integers m.

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- For a field $\mathbb{F}$ and a square matrix $A \in \mathbb{F}^{n \times n}$, we define
  - $A^0 := I_n$;
  - $A^{m+1} := A^m A$ for all non-negative integers m.
- So, by convention, we set $A^0 := I_n$, and for any positive integer m, we have that

$$A^m = \underbrace{A \dots A}_{m},$$

where we did not have to indicate parentheses since, by Theorem 1.7.5, matrix multiplication is associative.

2. The transpose of a matrix

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- Given a matrix $A \in \mathbb{F}^{n \times m}$ (where $\mathbb{F}$ is a field), the *transpose* of A, denoted by A^T, is the matrix in $\mathbb{F}^{m \times n}$ s.t. the i, j-th entry of A^T is the j, i-th entry of A, for all indices $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

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- In other words, to form A^T from A, the columns of A (from left to right) become the rows of A^T (from top to bottom), and likewise, the rows of A (from top to bottom) become the columns of A^T (from left to right).

$$A = \begin{bmatrix} {\color{red}\blacklozenge} & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}\lozenge} \\ {\color{blue}\blacklozenge} & {\color{blue}*} & {\color{blue}*} & {\color{blue}*} & {\color{blue}*} & {\color{blue}\lozenge} \\ {\color{green}\blacklozenge} & {\color{green}*} & {\color{green}*} & {\color{green}*} & {\color{green}*} & {\color{green}\lozenge} \\ {\color{purple}\blacklozenge} & {\color{purple}*} & {\color{purple}*} & {\color{purple}*} & {\color{purple}*} & {\color{purple}\lozenge} \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} {\color{red}\blacklozenge} & {\color{blue}\blacklozenge} & {\color{green}\blacklozenge} & {\color{purple}\blacklozenge} \\ {\color{red}*} & {\color{blue}*} & {\color{green}*} & {\color{purple}*} \\ {\color{red}*} & {\color{blue}*} & {\color{green}*} & {\color{purple}*} \\ {\color{red}*} & {\color{blue}*} & {\color{green}*} & {\color{purple}*} \\ {\color{red}*} & {\color{blue}*} & {\color{green}*} & {\color{purple}*} \\ {\color{red}\lozenge} & {\color{blue}\lozenge} & {\color{green}\lozenge} & {\color{purple}\lozenge} \end{bmatrix}$$

$$A = \begin{bmatrix} \blacklozenge & * & * & * & * & \lozenge \\ \blacklozenge & * & * & * & * & \lozenge \\ \blacklozenge & * & * & * & * & \lozenge \\ \blacklozenge & * & * & * & * & \lozenge \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \lozenge & \lozenge & \lozenge & \lozenge \end{bmatrix}$$

- For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

- In order to save space, we often specify column vectors in terms of transposes of row vectors.
- For instance, we often write something like

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}^T \text{ instead of } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Proposition 1.8.1

For any matrices A and B, and any scalar α, the following hold (provided the matrices are of compatible size for the operation in question, and the entries of our matrices and our scalar belong to the same field $\mathbb{F}$):

a. $(A^T)^T = A$;
b. $(A+B)^T = A^T + B^T$;
c. $(\alpha A)^T = \alpha A^T$
d. $(AB)^T = B^T A^T$.

Proof.

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Proof. Parts (a), (b), and (c) are obvious. Let us prove (d). Fix matrices $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times p}$, and set $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times m}$ and $B = \begin{bmatrix} b_{i,j} \end{bmatrix}_{m \times p}$.

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- (d) $(AB)^T = B^T A^T$.

Proof. Parts (a), (b), and (c) are obvious. Let us prove (d). Fix matrices $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times p}$, and set $A = \left[\begin{array}{c} a_{i,j} \end{array}\right]_{n\times m}$ and $B = \left[\begin{array}{c} b_{i,j} \end{array}\right]_{m\times p}$. Clearly, $AB \in \mathbb{F}^{n\times p}$, and so $(AB)^T \in \mathbb{F}^{p\times n}$. On the other hand, we have that $B^T \in \mathbb{F}^{p\times m}$ and $A^T \in \mathbb{F}^{m\times n}$, and so $B^T A^T \in \mathbb{F}^{p\times n}$. So, both $(AB)^T$ and $B^T A^T$ are $p \times n$ matrices with entries in $\mathbb{F}$. It remains to show that the corresponding entries of $(AB)^T$ and $B^T A^T$ are the same.

(d) $(AB)^T = B^T A^T$

Proof of (d) (continued).

(d) $(AB)^T = B^T A^T$

Proof of (d) (continued). Fix indices $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, n\}$; we will show that the i, j-th entry of $(AB)^T$ is equal to the i, j-th entry of $B^T A^T$.

By the definition of matrix transpose, the i, j-th entry of $(AB)^T$ is equal to the j, i-th entry of AB, which is equal to $\sum_{k=1}^{m} a_{j,k} b_{k,i}$.

(d) $(AB)^T = B^T A^T$

Proof of (d) (continued). Fix indices $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, n\}$; we will show that the i, j-th entry of $(AB)^T$ is equal to the i, j-th entry of $B^T A^T$.

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We now compute the i, j-th entry of $B^T A^T$. We observe that i-th row of the matrix B^T is $\begin{bmatrix} b_{1,i} & b_{2,i} & \dots & b_{m,i} \end{bmatrix}$, whereas the j-th column of the matrix A^T is $\begin{bmatrix} a_{j,1} & a_{j,2} & \dots & a_{j,m} \end{bmatrix}^T$.

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$b_{1,i} a_{j,1} + b_{2,i} a_{j,2} + \dots + b_{m,i} a_{j,m} = \sum\limits_{k=1}^{m} b_{k,i} a_{j,k} = \sum\limits_{k=1}^{m} a_{j,k} b_{k,i}$.

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Proof of (d) (continued). Fix indices $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, n\}$; we will show that the i,j-th entry of $(AB)^T$ is equal to the i,j-th entry of $B^T A^T$.

By the definition of matrix transpose, the i,j-th entry of $(AB)^T$ is equal to the j,i-th entry of AB, which is equal to $\sum_{k=1}^{m} a_{j,k} b_{k,i}$.

We now compute the i,j-th entry of $B^T A^T$. We observe that i-th row of the matrix B^T is $\begin{bmatrix} b_{1,i} & b_{2,i} & \dots & b_{m,i} \end{bmatrix}$, whereas the j-th column of the matrix A^T is $\begin{bmatrix} a_{j,1} & a_{j,2} & \dots & a_{j,m} \end{bmatrix}^T$. So, the i,j-th entry of the matrix $B^T A^T$ is

$$b_{1,i} a_{j,1} + b_{2,i} a_{j,2} + \dots + b_{m,i} a_{j,m} = \sum_{k=1}^{m} b_{k,i} a_{j,k} = \sum_{k=1}^{m} a_{j,k} b_{k,i}.$$

We have now shown that the corresponding entries of the $p \times n$ matrices $(AB)^T$ and $B^T A^T$ are the same, and we deduce that $(AB)^T = B^T A^T$. This proves (d). $\square$

Proposition 1.8.1

For any matrices A and B, and any scalar α, the following hold (provided the matrices are of compatible size for the operation in question, and the entries of our matrices and our scalar belong to the same field $\mathbb{F}$):

- (a) $(A^T)^T = A$;
- (b) $(A+B)^T = A^T + B^T$;
- (c) $(\alpha A)^T = \alpha A^T$
- (d) $(AB)^T = B^T A^T$.

3. Solving matrix equations of the form $AX = B$ and $XA = B$

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Example 1.9.1

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the matrix equation $AX = B$.[a] How many solutions does the equation $AX = B$ have?

[a] Note that solutions of the matrix equation $AX = B$ are 4×2 real matrices.

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with entries understood to be in $\mathbb{R}$. Solve the matrix equation $AX = B$.[a] How many solutions does the equation $AX = B$ have?

[a] Note that solutions of the matrix equation $AX = B$ are 4×2 real matrices.

- We give two solutions.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix},\ B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix}, \text{ solve } AX = B.$$

Solution#1.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix},\ B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix},\ \text{solve } AX = B.$$

Solution#1. Set $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$.

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Solution#1. Set $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$. Then $AX = \begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{bmatrix}$, and so the equation $AX = B$ is equivalent to

$$\begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}.$$

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Solution#1. Set $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$. Then $AX = \begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{bmatrix}$, and so the equation $AX = B$ is equivalent to

$$\begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}.$$

So, we need to solve two matrix-vector equations, namely $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$. We solve these two equations one by one.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix},\ B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix}, \text{ solve } AX = B.$$

Solution#1 (continued). First, we solve the matrix-vector equation $A\mathbf{x}_1 = \mathbf{b}_1$.

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix}$, solve $AX = B$.

Solution#1 (continued). First, we solve the matrix-vector equation $A\mathbf{x}_1 = \mathbf{b}_1$. We form the augmented matrix $\left[\begin{array}{c:c} A & \mathbf{b}_1 \end{array}\right]$ and we row reduce to obtain its reduced row echelon form:

$$\left[\begin{array}{c:c} A & \mathbf{b}_1 \end{array}\right] = \left[\begin{array}{rrrr:r} 1 & 2 & 3 & 4 & 5 \\ -1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 0 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{rrrr:r} 1 & 0 & 0 & \frac{31}{4} & \frac{35}{4} \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & -\frac{13}{4} & -\frac{13}{4} \end{array}\right].$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix},\ B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix}, \text{ solve } AX = B.$$

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We now read off the solutions for $\mathbf{x}_1$:

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{31}{4}s + \frac{35}{4} \\ -3s + 3 \\ \frac{13}{4}s - \frac{13}{4} \\ s \end{bmatrix}, \quad \text{where } s \in \mathbb{R}.$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix}, \text{ solve } AX = B.$$

Solution#1 (continued). We now solve the matrix-vector equation $A\mathbf{x}_2 = \mathbf{b}_2$.

$$A = \left[\begin{array}{rrrr} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{array}\right],\ B = \left[\begin{array}{rr} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{array}\right], \text{ solve } AX = B.$$

Solution#1 (continued). We now solve the matrix-vector equation $A\mathbf{x}_2 = \mathbf{b}_2$. We form the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b}_2 \end{array}\right]$ and we row reduce to obtain its reduced row echelon form:

$$\left[\begin{array}{c|c} A & \mathbf{b}_2 \end{array}\right] = \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 3 \\ -1 & 3 & 1 & -2 & 1 \\ 0 & 1 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & \frac{31}{4} & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -\frac{13}{4} & 1 \end{array}\right].$$

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix}$, solve $AX = B$.

Solution#1 (continued). We now solve the matrix-vector equation $A\mathbf{x}_2 = \mathbf{b}_2$. We form the augmented matrix $\left[\begin{array}{c:c} A & \mathbf{b}_2 \end{array}\right]$ and we row reduce to obtain its reduced row echelon form:

$$\left[\begin{array}{c:c} A & \mathbf{b}_2 \end{array}\right] = \left[\begin{array}{rrrr:r} 1 & 2 & 3 & 4 & 3 \\ -1 & 3 & 1 & -2 & 1 \\ 0 & 1 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr:r} 1 & 0 & 0 & \frac{31}{4} & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -\frac{13}{4} & 1 \end{array}\right].$$

We now read off the solutions for $\mathbf{x}_2$:

$$\mathbf{x}_2 = \begin{bmatrix} -\frac{31}{4}t \\ \\ -3t \\ \\ \frac{13}{4}t + 1 \\ \\ t \end{bmatrix}, \quad \text{where } t \in \mathbb{R}.$$

Solution#1 (continued). We now read off the general solution for $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$:

$$X = \begin{bmatrix} -\frac{31}{4}s + \frac{35}{4} & -\frac{31}{4}t \\ -3s + 3 & -3t \\ \frac{13}{4}s - \frac{13}{4} & \frac{13}{4}t + 1 \\ s & t \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

There are two parameters (namely, s and t), and they can each take infinitely many values (because $\mathbb{R}$ is infinite). So, the equation $AX = B$ has infinitely many solutions. $\square$

Solution#1 (continued). We now read off the general solution for $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$:

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There are two parameters (namely, s and t), and they can each take infinitely many values (because $\mathbb{R}$ is infinite). So, the equation $AX = B$ has infinitely many solutions. $\square$

Remark: Note that the parameters (namely, s and t) from the solution above are different for different columns! This is because the equations $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$ are solved independently, and so the parameter that appears in $\mathbf{x}_1$ is independent of the one that appears in $\mathbf{x}_2$.

Remark:

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- We had to solve a separate matrix-vector equation for each column of B, and each of these matrix-vector equations involved forming an augmented matrix and finding its reduced row echelon form.
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- Solution #1 is correct, but rather inefficient.
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  - Since B has two columns, this translated into two matrix-vector equations. In general, if B has m columns, we get m matrix-vector equations.
- Luckily, we can do better by essentially solving these two matrix-vector equations simultaneously.

Example 1.9.1

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the matrix equation $AX = B$.[a] How many solutions does the equation $AX = B$ have?

[a]Note that solutions of the matrix equation $AX = B$ are 4×2 real matrices.

Solution#2.

Example 1.9.1

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the matrix equation $AX = B$.[a] How many solutions does the equation $AX = B$ have?

[a] Note that solutions of the matrix equation $AX = B$ are 4×2 real matrices.

Solution#2. We first form the matrix $\left[\begin{array}{c:c} A & B \end{array}\right]$ and row reduce to find its reduced row echelon form.

$$\left[\begin{array}{c:c} A & B \end{array}\right] = \left[\begin{array}{cccc:cc} 1 & 2 & 3 & 4 & 5 & 3 \\ -1 & 3 & 1 & -2 & -3 & 1 \\ 0 & 1 & 0 & 3 & 3 & 0 \end{array}\right]$$

Solution#2 (continued). After row reducing, we obtain the following matrix:

$$\mathrm{RREF}\left(\left[\begin{array}{c:c} A & B \end{array}\right]\right) = \left[\begin{array}{cccc:cc} 1 & 0 & 0 & \frac{31}{4} & \color{red}{\frac{35}{4}} & \color{blue}{0} \\ 0 & 1 & 0 & 3 & \color{red}{3} & \color{blue}{0} \\ 0 & 0 & 1 & -\frac{13}{4} & \color{red}{-\frac{13}{4}} & \color{blue}{1} \end{array}\right].$$

Solution#2 (continued). After row reducing, we obtain the following matrix:

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We now read off the columns of X one by one.

Solution#2 (continued). After row reducing, we obtain the following matrix:

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We now read off the columns of X one by one.

- We read off the first column of X by reading off the solutions of the matrix-vector equation encoded by the matrix obtained by taking the submatrix to the left of the vertical dotted line, plus the first column to the right of the vertical dotted line (i.e. the red column) of $\text{RREF}\left(\left[\begin{array}{c:c} A & B \end{array}\right]\right)$.

Solution#2 (continued). After row reducing, we obtain the following matrix:

$$\mathrm{RREF}\left(\left[\begin{array}{c:c} A & B \end{array}\right]\right) = \left[\begin{array}{cccc:cc} 1 & 0 & 0 & \frac{31}{4} & \color{red}{\frac{35}{4}} & \color{blue}{0} \\ 0 & 1 & 0 & 3 & \color{red}{3} & \color{blue}{0} \\ 0 & 0 & 1 & -\frac{13}{4} & \color{red}{-\frac{13}{4}} & \color{blue}{1} \end{array}\right].$$

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- We read off the second column of X by reading off the solutions of the matrix-vector equation encoded by the matrix obtained by taking the submatrix to the left of the vertical dotted line, plus the second column to the right of the vertical dotted line (i.e. the blue column) of $\mathrm{RREF}\left(\left[\begin{array}{c:c} A & B \end{array}\right]\right)$.

Solution#2 (continued). Reminder:

$$\operatorname{RREF}\left(\left[\begin{array}{c:c} A & B \end{array}\right]\right) = \left[\begin{array}{cccc:cc} 1 & 0 & 0 & \frac{31}{4} & \color{red}{\frac{35}{4}} & \color{blue}{0} \\ 0 & 1 & 0 & 3 & \color{red}{3} & \color{blue}{0} \\ 0 & 0 & 1 & -\frac{13}{4} & \color{red}{-\frac{13}{4}} & \color{blue}{1} \end{array}\right].$$

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The solutions are as follows:

$$X = \begin{bmatrix} -\frac{31}{4}s + \frac{35}{4} & -\frac{31}{4}t \\ -3s + 3 & -3t \\ \frac{13}{4}s - \frac{13}{4} & \frac{13}{4}t + 1 \\ s & t \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

There are two parameters (namely, s and t), and they can each take infinitely many values (because $\mathbb{R}$ is infinite). So, the equation $AX = B$ has infinitely many solutions. $\square$

Recipe for solving matrix equations of the form $AX = B$.

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2. We check if $\mathrm{RREF}(\left[\begin{array}{c:c} A & B \end{array}\right])$ has a row of the form $\left[\begin{array}{ccc:ccc} 0 & \dots & 0 & * & \dots & * \end{array}\right]$, where at least one of the $*$'s (to the right of the vertical dotted line) is non-zero.

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    a. If such a row exists, then the matrix equation $AX = B$ is inconsistent (i.e. has no solutions).

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   1. (a) If such a row exists, then the matrix equation $AX = B$ is inconsistent (i.e. has no solutions).
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Recipe for solving matrix equations of the form $AX = B$. Suppose that A is an $n \times m$ matrix and B is an $n \times p$ matrix (both with entries in some field $\mathbb{F}$), and we wish to solve the matrix equation $AX = B$. We proceed as follows:

1. We form the $n \times (m+p)$ matrix $\left[\begin{array}{c:c} A & B \end{array}\right]$ and find its reduced row echelon form.
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   a. If such a row exists, then the matrix equation $AX = B$ is inconsistent (i.e. has no solutions).
   b. If no such row exits, then the matrix equation $AX = B$ is consistent (i.e. has at least one solution).
      - For each $k \in \{1, \dots, p\}$, we read off the k-th column of X by focusing on the part of $\text{RREF}\Big(\left[\begin{array}{c:c} A & B \end{array}\right]\Big)$ to the left of the vertical dotted line, plus the k-th column of $\text{RREF}\Big(\left[\begin{array}{c:c} A & B \end{array}\right]\Big)$ to the right of the vertical dotted line.

Recipe for solving matrix equations of the form $AX = B$.

Suppose that A is an $n \times m$ matrix and B is an $n \times p$ matrix (both with entries in some field $\mathbb{F}$), and we wish to solve the matrix equation $AX = B$. We proceed as follows:

1. We form the $n \times (m+p)$ matrix $\left[\begin{array}{c:c} A & B \end{array}\right]$ and find its reduced row echelon form.
2. We check if $\text{RREF}(\left[\begin{array}{c:c} A & B \end{array}\right])$ has a row of the form $\left[\begin{array}{ccc:ccc} 0 & \dots & 0 & * & \dots & * \end{array}\right]$, where at least one of the $*$'s (to the right of the vertical dotted line) is non-zero.
    a. If such a row exists, then the matrix equation $AX = B$ is inconsistent (i.e. has no solutions).
    b. If no such row exits, then the matrix equation $AX = B$ is consistent (i.e. has at least one solution).
        - For each $k \in \{1, \dots, p\}$, we read off the k-th column of X by focusing on the part of $\text{RREF}\big(\left[\begin{array}{c:c} A & B \end{array}\right]\big)$ to the left of the vertical dotted line, plus the k-th column of $\text{RREF}\big(\left[\begin{array}{c:c} A & B \end{array}\right]\big)$ to the right of the vertical dotted line.
        - If there are any free variables, remember to use different letters for the parameters in different columns, as in the solution of Example 1.9.1.

Example 1.9.2

Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 & 1 & 3 \\ 4 & 3 & 1 & 3 \\ 2 & 1 & 1 & 3 \\ 2 & 1 & 2 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the matrix equation $AX = B$.[a] How many solutions does the equation $AX = B$ have?

---

[a]Note that solutions of the matrix equation $AX = B$ are 3×4 real matrices.

Solution. We first form the matrix

$$\left[\begin{array}{c:c} A & B \end{array}\right] = \left[\begin{array}{rrr:rrrr} 1 & 1 & 1 & 4 & 3 & 1 & 3 \\ 1 & 1 & 1 & 4 & 3 & 1 & 3 \\ 1 & 2 & -1 & 2 & 1 & 1 & 3 \\ 1 & 2 & -1 & 2 & 1 & 2 & 3 \end{array}\right].$$

Solution. We first form the matrix

$$\left[\begin{array}{c:c} A & B \end{array}\right] = \left[\begin{array}{rrr:rrrr} 1 & 1 & 1 & 4 & 3 & 1 & 3 \\ 1 & 1 & 1 & 4 & 3 & 1 & 3 \\ 1 & 2 & -1 & 2 & 1 & 1 & 3 \\ 1 & 2 & -1 & 2 & 1 & 2 & 3 \end{array}\right].$$

After row reducing, we obtain

$$\mathrm{RREF}\left(\left[\begin{array}{c:c} A & B \end{array}\right]\right) = \left[\begin{array}{rrr:rrrr} 1 & 0 & 3 & 6 & 5 & 0 & 3 \\ 0 & 1 & -2 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Solution. We first form the matrix

$$\left[\begin{array}{c:c} A & B \end{array}\right] = \left[\begin{array}{rrr:rrrr} 1 & 1 & 1 & 4 & 3 & 1 & 3 \\ 1 & 1 & 1 & 4 & 3 & 1 & 3 \\ 1 & 2 & -1 & 2 & 1 & 1 & 3 \\ 1 & 2 & -1 & 2 & 1 & 2 & 3 \end{array}\right].$$

After row reducing, we obtain

$$\mathrm{RREF}\Big(\left[\begin{array}{c:c} A & B \end{array}\right]\Big) = \left[\begin{array}{rrr:rrrr} 1 & 0 & 3 & 6 & 5 & 0 & 3 \\ 0 & 1 & -2 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

By considering the third row of $\mathrm{RREF}(\left[\begin{array}{c:c} A & B \end{array}\right])$, we see that the matrix equation $AX = B$ is inconsistent, i.e. it has no solutions. $\square$

- What about matrix equations of the form $XA = B$?

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- Such an equation is equivalent to the equation $(XA)^T = B^T$, which is, in turn, equivalent to $A^T X^T = B^T$
  - We are using Proposition 1.8.1(d).

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  - We are using Proposition 1.8.1(d).
- We solve the equation $A^T X^T = B^T$ for X^T, and then we take the transpose of the solution(s) to obtain X.

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  - We are using Proposition 1.8.1(d).
- We solve the equation $A^T X^T = B^T$ for X^T, and then we take the transpose of the solution(s) to obtain X.

Example 1.9.5

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 5 & 1 & -2 \\ 4 & 3 & 1 & -1 \\ 2 & 4 & 0 & -2 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the matrix equation $XA = B$.[a] How many solutions does the equation $XA = B$ have?

[a] Note that solutions of the matrix equation $XA = B$ are 3×2 real matrices.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 5 & 5 & 1 & -2 \\ 4 & 3 & 1 & -1 \\ 2 & 4 & 0 & -2 \end{bmatrix},\ \text{solve } XA = B.$$

Solution.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 5 & 5 & 1 & -2 \\ 4 & 3 & 1 & -1 \\ 2 & 4 & 0 & -2 \end{bmatrix}, \text{ solve } XA = B.$$

Solution. Note that $XA = B$ iff $A^T X^T = B^T$.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 5 & 5 & 1 & -2 \\ 4 & 3 & 1 & -1 \\ 2 & 4 & 0 & -2 \end{bmatrix}, \text{ solve } XA = B.$$

Solution. Note that $XA = B$ iff $A^T X^T = B^T$. We first find all the matrices X^T that satisfy $A^T X^T = B^T$, and then we take the transpose to obtain all the matrices X that satisfy $XA = B$.

$A = \left[\begin{array}{rrrr} 1 & 2 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{array}\right]$, $B = \left[\begin{array}{rrrr} 5 & 5 & 1 & -2 \\ 4 & 3 & 1 & -1 \\ 2 & 4 & 0 & -2 \end{array}\right]$, solve $XA = B$.

Solution. Note that $XA = B$ iff $A^T X^T = B^T$. We first find all the matrices X^T that satisfy $A^T X^T = B^T$, and then we take the transpose to obtain all the matrices X that satisfy $XA = B$. First, we have

$$A^T = \left[\begin{array}{rr} 1 & 3 \\ 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{array}\right] \quad \text{and} \quad B^T = \left[\begin{array}{rrr} 5 & 4 & 2 \\ 5 & 3 & 4 \\ 1 & 1 & 0 \\ -2 & -1 & -2 \end{array}\right].$$

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 5 & 5 & 1 & -2 \\ 4 & 3 & 1 & -1 \\ 2 & 4 & 0 & -2 \end{bmatrix}, \text{ solve } XA = B.$$

Solution. Note that $XA = B$ iff $A^T X^T = B^T$. We first find all the matrices X^T that satisfy $A^T X^T = B^T$, and then we take the transpose to obtain all the matrices X that satisfy $XA = B$. First, we have

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 5 & 4 & 2 \\ 5 & 3 & 4 \\ 1 & 1 & 0 \\ -2 & -1 & -2 \end{bmatrix}.$$

We now form the matrix

$$\left[\begin{array}{c:c} A^T & B^T \end{array}\right] = \left[\begin{array}{cc:ccc} 1 & 3 & 5 & 4 & 2 \\ 2 & 1 & 5 & 3 & 4 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & -1 & -2 \end{array}\right],$$

$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 5 & 1 & -2 \\ 4 & 3 & 1 & -1 \\ 2 & 4 & 0 & -2 \end{bmatrix}$, solve $XA = B$.

Solution. Note that $XA = B$ iff $A^T X^T = B^T$. We first find all the matrices X^T that satisfy $A^T X^T = B^T$, and then we take the transpose to obtain all the matrices X that satisfy $XA = B$. First, we have

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 5 & 4 & 2 \\ 5 & 3 & 4 \\ 1 & 1 & 0 \\ -2 & -1 & -2 \end{bmatrix}.$$

We now form the matrix

$$\left[\, A^T \mid B^T \,\right] = \left[\begin{array}{cc|ccc} 1 & 3 & 5 & 4 & 2 \\ 2 & 1 & 5 & 3 & 4 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & -1 & -2 \end{array}\right],$$

and by row reducing, we obtain (next slide):

Solution (continued).

$$\mathrm{RREF}\Big(\left[\begin{array}{c:c} A^T & B^T \end{array}\right]\Big) = \left[\begin{array}{cc:ccc} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Solution (continued).

$$\text{RREF}\Big(\left[\begin{array}{c:c} A^T & B^T \end{array}\right] \Big) = \left[\begin{array}{cc:ccc} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using the matrix above, we can solve for X^T. There is only one solution, namely:

$$X^T = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution (continued).

$$\mathrm{RREF}\Big(\left[\begin{array}{c:c} A^T & B^T \end{array}\right]\Big) = \left[\begin{array}{cc:ccc} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using the matrix above, we can solve for X^T. There is only one solution, namely:

$$X^T = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Thus, the equation $XA = B$ has a unique solution, namely:

$$X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

(The number of solutions of the matrix equation $XA = B$ is one.) $\square$