Linear Algebra 1

Lecture #2

Matrix-vector equations. The rank of a matrix

Irena Penev

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1. Algebraic operations on vectors and linear span

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2. Matrix-vector multiplication

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1. Algebraic operations on vectors and linear span
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4. The rank of a matrix

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    - $\mathbf{x} + \mathbf{y} := \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$;
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1. Algebraic operations on vectors and linear span

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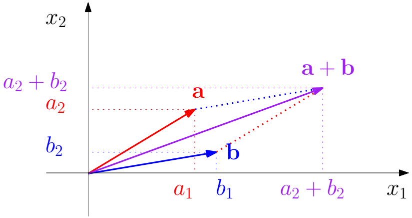
  - For a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{F}^n$ and a scalar $\alpha \in \mathbb{F}$, we define
    - $\alpha\mathbf{x} := \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$.

Example 1.4.1

Consider the vectors $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ in $\mathbb{Z}_3^4$. Then

- $\mathbf{x} + \mathbf{y} = \begin{bmatrix} 0+1 \\ 1+0 \\ 2+2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$;
- $2\mathbf{x} = \begin{bmatrix} 2\cdot 0 \\ 2\cdot 1 \\ 2\cdot 2 \\ 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$.

- Vector addition and scalar multiplication in $\mathbb{R}^2$ have a nice geometric interpretation.
- To add two vectors in $\mathbb{R}^2$, say $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, we apply the "parallelogram rule."



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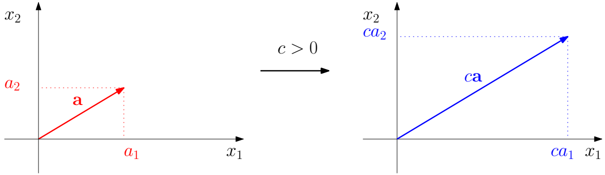

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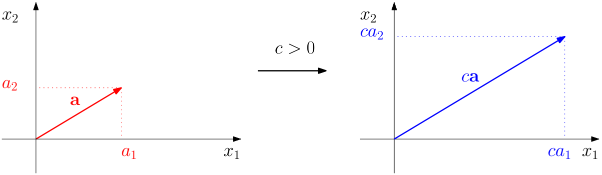

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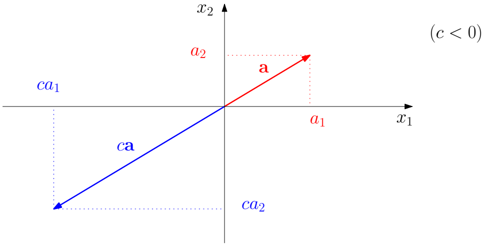

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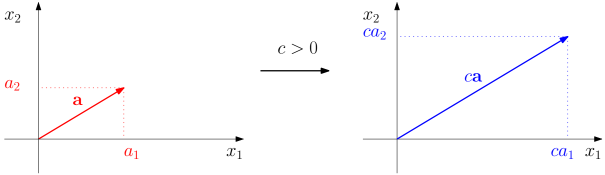

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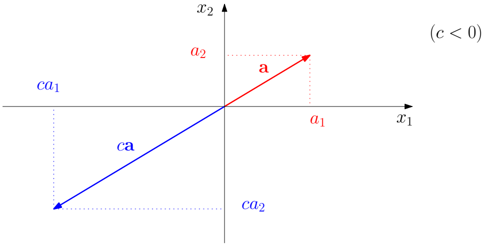
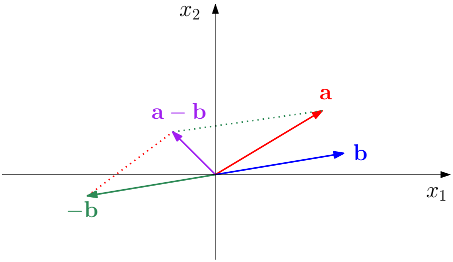


- If $c = 0$, then $c\mathbf{a} = \mathbf{0}$, which is simply the origin.

- For vectors $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\mathbb{R}^2$, we note that

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}.$$



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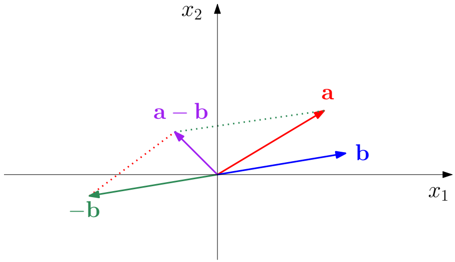

- For vectors in $\mathbb{R}^3$, we have a similar geometric interpretation of vector addition and scalar multiplication (and vector subtraction), only in the three-dimensional Euclidean space.

Definition

Suppose $\mathbb{F}$ is some field. A *linear combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{F}^n$ is any sum of the form

$$\sum_{i=1}^{k} \alpha_i \mathbf{v}_i \quad = \quad \alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k,$$

where $\alpha_1, \ldots, \alpha_k$ are scalars from the field $\mathbb{F}$.

- For example, in $\mathbb{R}^3$, vectors $\begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ -9 \\ -3 \end{bmatrix}$ are linear combinations of the vectors $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ because

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- Similarly, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a linear combination of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbb{Z}_3^2$ because $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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- We note that in $\mathbb{F}^n$ (where $\mathbb{F}$ is a field), the zero vector $\mathbf{0}$ is a linear combination of **any** vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ because

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- Moreover, we define the "empty sum" of vectors in $\mathbb{F}^n$ (or the sum of an "empty list" of vectors in $\mathbb{F}^n$) to be $\mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathbb{F}^n$.

Definition

The *linear span* (or simply *span*) of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{F}^n$ (where $\mathbb{F}$ is a field), denoted by $\text{Span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_k\})$ or simply $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$, is the set of all linear combinations of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. In other words,

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- Obviously, $\mathrm{Span}(\mathbf{0}) = \{\mathbf{0}\}$.

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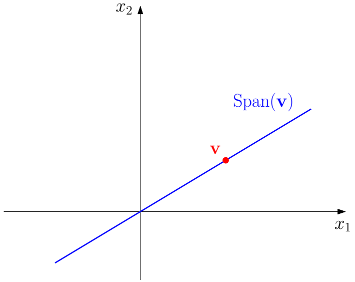
- As we discussed above, $\text{Span}(\emptyset) = \{\mathbf{0}\}$ and $\text{Span}(\mathbf{0}) = \{\mathbf{0}\}$.
- If $\mathbf{v} \neq \mathbf{0}$, then $\text{Span}(\mathbf{v}) = \{\alpha\mathbf{v} \mid \alpha \in \mathbb{R}\}$ is the line through the origin containing $\mathbf{v}$: indeed, $\text{Span}(\mathbf{v})$ is the set of all scalar multiples of $\mathbf{v}$, which is precisely the line through $\mathbf{0}$ and $\mathbf{v}$.



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- If neither of those vectors is a scalar multiple of the other (and in particular, neither of the two vectors is $\mathbf{0}$), then $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ is the plane through $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2$.

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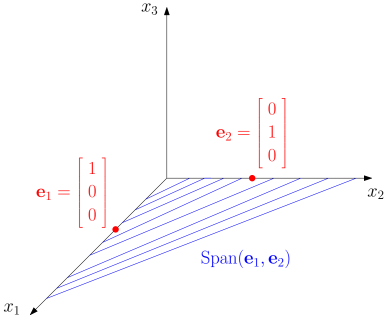
- The case that is particularly easy to visualize is that of the vectors $\mathbf{e}_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$:

$$\begin{aligned} \text{Span}(\mathbf{e}_1, \mathbf{e}_2) &= \left\{ a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \mid a_1, a_2 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R} \right\}, \end{aligned}$$

which is simply the x_1x_2-plane in $\mathbb{R}^3$ (next slide).



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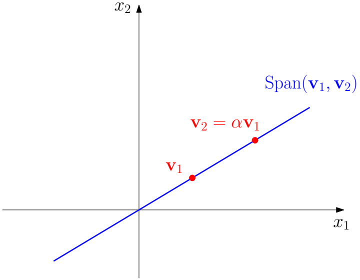

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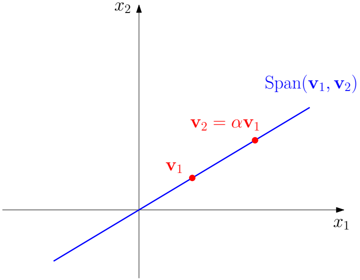

- In general, for vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$, the set Span$(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is the smallest "flat" (point, line, plane, or higher dimensional generalization) containing the **origin** and all the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

2. Matrix-vector multiplication

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Definition

Suppose that $\mathbb{F}$ is some field. Given a matrix $A \in \mathbb{F}^{n \times m}$ and a vector $\mathbf{x} \in \mathbb{F}^m$, say

$$A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

we define the *matrix-vector product* $A\mathbf{x}$ as follows:

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$$A\mathbf{x} := \sum_{i=1}^{m} x_i \mathbf{a}_i = x_1 \mathbf{a}_1 + \cdots + x_m \mathbf{a}_m.$$

- Thus, $A\mathbf{x}$ is a linear combination of the columns of A, and the weights/scalars in front of the columns are determined by the entries of the vector $\mathbf{x}$.

- **Reminder:** For a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix} \mathbb{F}^{n \times m}$ and vector $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ in $\mathbb{F}^m$ (where $\mathbb{F}$ is a field):

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  - entries of the matrix A and entries of the vector $\mathbf{x}$ must belong to the same field;
  - the number of **columns** of A must be the same as the number of **entries** of $\mathbf{x}$.
- Schematically, we have the following:

$$\underbrace{A}_{\in\mathbb{F}^{n\times m}} \; \underbrace{\mathbf{x}}_{\in\mathbb{F}^{m}} \; = \; \underbrace{A\mathbf{x}}_{\in\mathbb{F}^{n}}.$$

Example 1.4.2

Consider the matrix $A \in \mathbb{R}^{3\times 2}$ and vector $\mathbf{x} \in \mathbb{R}^2$, given below:

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 0 \\ 3 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Then

$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} -1 & 2 \\ 2 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= 2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}. \end{aligned}$$

Example 1.4.3

Consider the matrix $A \in \mathbb{Z}_2^{2\times 3}$ and vector $\mathbf{x} \in \mathbb{Z}_2^3$, given below:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

- **Remark:** Suppose that $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array} \right]$ is a matrix in $\mathbb{F}^{n \times m}$ (where $\mathbb{F}$ is some field).

- **Remark:** Suppose that $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ is a matrix in $\mathbb{F}^{n \times m}$ (where $\mathbb{F}$ is some field). Then

$$\begin{aligned} \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) &= \Big\{ x_1 \mathbf{a}_1 + \dots + x_m \mathbf{a}_m \mid x_1, \dots, x_m \in \mathbb{F} \Big\} \\ &= \left\{ \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \mid x_1, \dots, x_m \in \mathbb{F} \right\} \\ &= \Big\{ A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m \Big\}. \end{aligned}$$

- **Remark:** Suppose that $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ is a matrix in $\mathbb{F}^{n \times m}$ (where $\mathbb{F}$ is some field). Then

$$\begin{aligned} \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) &= \Big\{ x_1\mathbf{a}_1 + \dots + x_m\mathbf{a}_m \mid x_1, \dots, x_m \in \mathbb{F} \Big\} \\ &= \Big\{ \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \mid x_1, \dots, x_m \in \mathbb{F} \Big\} \\ &= \Big\{ A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m \Big\}. \end{aligned}$$

- So, $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$, which we defined as the set of all linear combinations of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$, is in fact the set of all possible matrix-vector products $A\mathbf{x}$ (where our matrix $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ is fixed, and the vector $\mathbf{x} \in \mathbb{F}^m$ is allowed to vary).

- **Remark:** Suppose that $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ is a matrix in $\mathbb{F}^{n \times m}$ (where $\mathbb{F}$ is some field). Then

$$\begin{aligned} \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) &= \Big\{ x_1\mathbf{a}_1 + \dots + x_m\mathbf{a}_m \mid x_1, \dots, x_m \in \mathbb{F} \Big\} \\ &= \Big\{ \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \mid x_1, \dots, x_m \in \mathbb{F} \Big\} \\ &= \Big\{ A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^m \Big\}. \end{aligned}$$

- So, $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$, which we defined as the set of all linear combinations of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$, is in fact the set of all possible matrix-vector products $A\mathbf{x}$ (where our matrix $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ is fixed, and the vector $\mathbf{x} \in \mathbb{F}^m$ is allowed to vary).
- We note that $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$, the span of the columns of A, has a special name: it is called the "column space" of the matrix A, and it is denoted by $\text{Col}(A)$.
    - To be studied later in the course.

- Let $\mathbb{F}$ be a field. For each positive integer n and index $i \in \{1, \dots, n\}$, the vector $\mathbf{e}_i^n$ is the vector in $\mathbb{F}^n$ whose i-th entry is 1, and all of whose other entries are 0's.

$$\mathbf{e}_i^n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longleftarrow i\text{-th entry}$$

- When n is clear from context, we drop the superscript n, and we write $\mathbf{e}_1, \dots, \mathbf{e}_n$ instead of $\mathbf{e}_1^n, \dots, \mathbf{e}_n^n$, respectively.
- Vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are called the *standard basis vectors* of $\mathbb{F}^n$, and the set $\mathcal{E}_n := \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is called the *standard basis* of $\mathbb{F}^n$.

$$\mathbf{e}_i^n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \color{red}{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \color{red}{\longleftarrow i\text{-th entry}}$$

- We note that any vector $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{F}^n$ can be expressed as a linear combination of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in a unique way, namely

$$\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n.$$

Proposition 1.4.4

Let $\mathbb{F}$ be a field, and let $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{array}\right]$ be a matrix in $\mathbb{F}^{n \times m}$. Then for all indices $i \in \{1, \dots, m\}$, we have that $A\mathbf{e}_i^m = \mathbf{a}_i$.

- **Remark:** Proposition 1.4.4 states that multiplying a matrix by the i-th standard basis vector yields the i-th column of the matrix that we started with.

Proposition 1.4.4

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Proof. Fix $i \in \{1, \dots, m\}$. Then

$$A\mathbf{e}_i^m = \begin{bmatrix} \mathbf{a}_1 & \dots \mathbf{a}_{i-1} & \mathbf{a}_i & \mathbf{a}_{i+1} & \dots & \mathbf{a}_m \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longleftarrow i\text{-th entry}$$

$$= 0\mathbf{a}_1 + \dots + 0\mathbf{a}_{i-1} + 1\mathbf{a}_i + 0\mathbf{a}_{i+1} + \dots + 0\mathbf{a}_m = \mathbf{a}_i,$$

which is what we needed to show. □

- For a field $\mathbb{F}$, the *identity matrix* in $\mathbb{F}^{n\times n}$ is the $n \times n$ matrix

$$I_n \quad := \quad \left[\begin{array}{ccc} \mathbf{e}_1^n & \dots & \mathbf{e}_n^n \end{array} \right].$$

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$$I_n \quad := \quad \left[\begin{array}{ccc} \mathbf{e}_1^n & \dots & \mathbf{e}_n^n \end{array} \right].$$

- In other words, the identity matrix I_n is the $n \times n$ matrix with 1's on the main diagonal and 0's elsewhere (where the 1's and the 0's are from the field $\mathbb{F}$). Schematically, we have that

$$I_n \quad = \quad \left[\begin{array}{cccccc} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{array} \right]_{n \times n}$$

for all positive integers n.

Proposition 1.4.5

Let $\mathbb{F}$ be a field. Then for all vectors $\mathbf{v} \in \mathbb{F}^n$, we have that $I_n \mathbf{v} = \mathbf{v}$.

- **Remark:** Proposition 1.4.5 states that if we multiply the identity matrix by a vector, we obtain that same vector.

For any vector $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{F}^n$, we have that (next slide):

$$
\begin{aligned}
I_n\mathbf{v} &= \begin{bmatrix} \mathbf{e}_1^n & \mathbf{e}_2^n & \dots & \mathbf{e}_n^n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= v_1\mathbf{e}_1^n + v_2\mathbf{e}_2^n + \cdots + v_n\mathbf{e}_n^n \\
&= v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \mathbf{v}. \qquad \square
\end{aligned}
$$

Proposition 1.4.4

Let $\mathbb{F}$ be a field, and let $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ be a matrix in $\mathbb{F}^{n \times m}$. Then for all indices $i \in \{1, \dots, m\}$, we have that $A\mathbf{e}_i^m = \mathbf{a}_i$.

Proposition 1.4.5

Let $\mathbb{F}$ be a field. Then for all vectors $\mathbf{v} \in \mathbb{F}^n$, we have that $I_n\mathbf{v} = \mathbf{v}$.

Proposition 1.4.4

Let $\mathbb{F}$ be a field, and let $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ be a matrix in $\mathbb{F}^{n \times m}$. Then for all indices $i \in \{1, \dots, m\}$, we have that $A\mathbf{e}_i^m = \mathbf{a}_i$.

Proposition 1.4.5

Let $\mathbb{F}$ be a field. Then for all vectors $\mathbf{v} \in \mathbb{F}^n$, we have that $I_n \mathbf{v} = \mathbf{v}$.

Proposition 1.4.6

Let $\mathbb{F}$ be a field. Then both the following hold:

- (a) for all $\mathbf{v} \in \mathbb{F}^m$, we have that $O_{n \times m}\mathbf{v} = \mathbf{0}$;[a]
- (b) for all matrices $A \in \mathbb{F}^{n \times m}$, we have that $A\mathbf{0} = \mathbf{0}$.[b]

---

[a] Here, the zero vector $\mathbf{0}$ belongs to $\mathbb{F}^n$.

[b] In the expression $A\mathbf{0} = \mathbf{0}$, we have that $\mathbf{0} \in \mathbb{F}^m$ and $\mathbf{0} \in \mathbb{F}^n$.

Proof. This readily follows from the definition of matrix-vector multiplication.

3. Matrix-vector equations

- A *matrix-vector equation* is an equation of the form

$$A\mathbf{x} = \mathbf{b},$$

where the matrix A and vector $\mathbf{b}$ are known, and the vector $\mathbf{x}$ is unknown.

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    - Moreover, the number of **rows** of A must be the same as the number of **entries** of $\mathbf{b}$.

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- A matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is equivalent to a system of linear equations whose augmented matrix is $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$.

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- A matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is equivalent to a system of linear equations whose augmented matrix is $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$.
    - Details: next slide.
    - The matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ will also be referred to as the *augmented matrix* of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.

$$
A\mathbf{x} = \mathbf{b} \iff \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}}_{=\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{=\mathbf{b}}
$$

$$
\iff x_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,2} \end{bmatrix} + \cdots + x_m \begin{bmatrix} a_{1,m} \\ a_{2,m} \\ \vdots \\ a_{n,m} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

$$
\iff \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

$$
\iff \left\{ \begin{array}{ccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots + a_{1,m}x_m & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots + a_{2,m}x_m & = & b_2 \\ & & & & & \vdots & \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \cdots + a_{n,m}x_m & = & b_n \end{array} \right.
$$

Example 1.5.1

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. How many solutions does the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ have?

Solution.

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Solution. The augmented matrix of $A\mathbf{x} = \mathbf{b}$ is

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cc:c} 1 & 2 & 2 \\ 3 & 6 & 6 \end{array}\right].$$

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Solution. The augmented matrix of $A\mathbf{x} = \mathbf{b}$ is

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 3 & 6 & 6 \end{array}\right].$$

We now row reduce in order to find $\text{RREF}\left(\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]\right)$, as follows:

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 3 & 6 & 6 \end{array}\right] \overset{R_3 \to R_3 - 3R_1}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

Solution (continued). The last matrix from the computation above is in reduced row echelon form, and we deduce that

$$\mathrm{RREF}\Big(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] \Big) \quad = \quad \left[\begin{array}{cc:c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

Solution (continued). The last matrix from the computation above is in reduced row echelon form, and we deduce that

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The matrix $\text{RREF}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right])$ is the augmented matrix of the linear system below.

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 2 \\ & & 0 & = & 0 \end{array}$$

Solution (continued). The last matrix from the computation above is in reduced row echelon form, and we deduce that

$$\mathrm{RREF}\Big(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]\Big) = \left[\begin{array}{cc:c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

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$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 2 \\ & & 0 & = & 0 \end{array}$$

The system is consistent, with one free variable (namely, x_2). We read off the solutions as follows.

$$\begin{array}{rcl} x_1 & = & -2s+2 \\ x_2 & = & s, \end{array} \qquad \text{where } s \in \mathbb{R}.$$

Solution (continued). So, the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} \quad = \quad \begin{bmatrix} -2s + 2 \\ s \end{bmatrix}, \qquad \text{where } s \in \mathbb{R}.$$

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Here is another way to write the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$:

$$\mathbf{x} \quad = \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \text{where } s \in \mathbb{R}.$$

Solution (continued). So, the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

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Here is another way to write the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$:

$$\mathbf{x} \;=\; \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \text{where } s \in \mathbb{R}.$$

The set of solutions of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\left\{ \begin{bmatrix} -2s+2 \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\} \;=\; \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid s \in \mathbb{R} \right\}.$$

Solution (continued). So, the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

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The set of solutions of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\left\{ \begin{bmatrix} -2s+2 \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\} \;=\; \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid s \in \mathbb{R} \right\}.$$

Since the parameter s can take infinitely many values (because $\mathbb{R}$ is infinite), the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. □

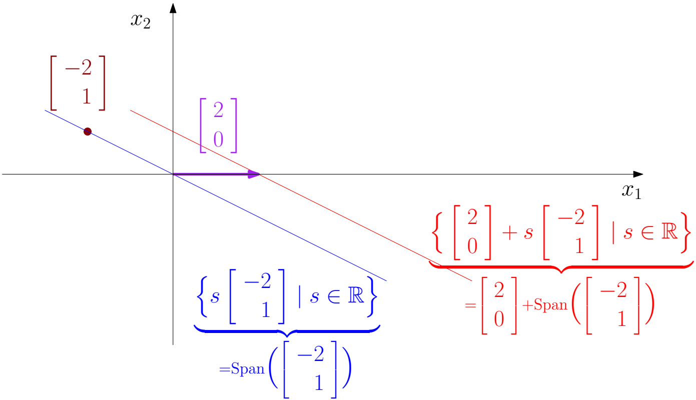
- **Reminder:** The solution set is

$$\left\{ \begin{bmatrix} -2s+2 \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid s \in \mathbb{R} \right\}.$$

- **Reminder:** The solution set is

$$\left\{ \begin{bmatrix} -2s+2 \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid s \in \mathbb{R} \right\}.$$

- This solution set has a geometric interpretation:



Example 1.5.2

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. How many solutions does the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ have?

Solution.

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with entries understood to be in $\mathbb{Z}_3$. How many solutions does the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ have?

Solution. The augmented matrix of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right].$$

Example 1.5.2

Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. How many solutions does the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ have?

Solution. The augmented matrix of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right].$$

We now row reduce in order to find $\text{RREF}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right])$, as follows (next slide):

Solution (continued).

$$
\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right]
$$

$$
\underset{\sim}{\substack{R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{array}\right]
$$

$$
\underset{\sim}{R_3 \to R_3 + 2R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]
$$

$$
\underset{\sim}{R_3 \to 2R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
\underset{\sim}{R_1 \to R_1 + R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
\underset{\sim}{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].
$$

Solution (continued). The last matrix from the computation above is in reduced row echelon form, and we deduce that

$$\text{RREF}\Big(\Big[\begin{array}{c:c} A & \mathbf{b} \end{array}\Big]\Big) = \left[\begin{array}{cccc:c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Solution (continued). The last matrix from the computation above is in reduced row echelon form, and we deduce that

$$\text{RREF}\Big(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]\Big) = \left[\begin{array}{cccc:c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

We see from $\text{RREF}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right])$ that the rightmost column of $\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]$ is a pivot column; consequently, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, i.e. the solution set of the equation $A\mathbf{x} = \mathbf{b}$ is $\emptyset$. (The number of solutions of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is zero.) $\square$

Solution (continued). The last matrix from the computation above is in reduced row echelon form, and we deduce that

$$\text{RREF}\Big(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]\Big) \quad = \quad \left[\begin{array}{cccc:c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

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- Here is the row reduction again (next slide):

$$
\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right]
$$

$$
\underset{\substack{R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1}}{\sim} \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{array}\right]
$$

$$
\underset{R_3 \to R_3 + 2R_2}{\sim} \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]
$$

$$
\underset{R_3 \to 2R_3}{\sim} \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
\underset{R_1 \to R_1 + R_3}{\sim} \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
\underset{R_1 \to R_1 + R_2}{\sim} \left[\begin{array}{cccc:c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].
$$

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] \sim \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

- We could in fact have stopped as soon as we got the red matrix (despite the fact that this matrix is not in reduced row echelon form).

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] \sim \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

- We could in fact have stopped as soon as we got the red matrix (despite the fact that this matrix is not in reduced row echelon form).
- This is because the bottom row of the red matrix encodes the equation $0 = 2$, which is has no solutions.

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] \sim \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

- We could in fact have stopped as soon as we got the red matrix (despite the fact that this matrix is not in reduced row echelon form).
- This is because the bottom row of the red matrix encodes the equation $0 = 2$, which is has no solutions.
- Indeed, as soon as we obtain a row of the form $\left[\begin{array}{ccc:c} 0 & \dots & 0 & \blacksquare \end{array}\right]$, where $\blacksquare$ is a non-zero number, we can stop row reducing, and we can deduce that the system has no solutions (because this row encodes the equation $0 = \blacksquare$, and $\blacksquare$ is non-zero).

4. The rank of a matrix

- The *rank* of a matrix A (with entries in some field $\mathbb{F}$), denoted by $\operatorname{rank}(A)$, is the number of pivot columns of A.
- Equivalently, $\operatorname{rank}(A)$ is the number of pivot positions of A, or the number of non-zero rows of any row echelon form of A.
- To find the rank of a matrix, we first find some row echelon form of that matrix (e.g. by performing the forward phase of row reduction; the backward phase is optional), and we count the number of pivot columns (or alternatively, the number of pivot positions, or the number of non-zero rows) of that row echelon matrix.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1.6.1

Find the rank of each of the following matrices.

(a) $A = \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix}$, with entries understood to be in $\mathbb{R}$;

(b) $B = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 1 \end{bmatrix}$, with entries understood to be in $\mathbb{Z}_3$.

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Solution#1. (a) In Example 1.3.9, we computed

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -3 & 7 & 0 & 4 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

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The matrix RREF(A) has three pivot columns (equivalently: three pivot positions or three non-zero rows), and so $\text{rank}(A) = 3$. □

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Solution#2.

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Solution#2. (a) In Example 1.3.9, we saw that the matrix A is row equivalent to the following matrix in row echelon form:

$$\begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

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This row echelon matrix has three pivot columns (equivalently: three pivot positions or three non-zero rows), and so $\text{rank}(A) = 3$. $\square$

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Find the rank of each of the following matrices.

(b) $B = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 1 \end{bmatrix}$, with entries understood to be in $\mathbb{Z}_3$.

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Example 1.6.1

Find the rank of each of the following matrices.

(b) $B = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 1 \end{bmatrix}$, with entries understood to be in $\mathbb{Z}_3$.

Solution#1. (b) In Example 1.3.10, we computed

$$\text{RREF}(B) = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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Find the rank of each of the following matrices.

- (b) $B = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 1 \end{bmatrix}$, with entries understood to be in $\mathbb{Z}_3$.

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$$\text{RREF}(B) = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix RREF(B) has three pivot columns (equivalently: three pivot positions or three non-zero rows), and so rank(B) $= 3$. □

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Solution#2. (b) In Example 1.3.10, we saw that the matrix B is row equivalent to the following matrix in row echelon form:

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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This row echelon matrix has three pivot columns (equivalently: three pivot positions or three non-zero rows), and so $\text{rank}(B) = 3$. $\square$

Corollary 1.3.8

Two matrices (with entries in some field) are row equivalent iff they have the same reduced row echelon form.

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Row equivalent matrices (with entries in some field) have the same rank.

Proof. Fix row equivalent matrices A and B (with entries in some field). By the definition of rank, $\text{rank}(A)$ is equal to the number of pivot columns of A, which is precisely the number of pivot columns of $\text{RREF}(A)$.

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Proof. Fix row equivalent matrices A and B (with entries in some field). By the definition of rank, $\text{rank}(A)$ is equal to the number of pivot columns of A, which is precisely the number of pivot columns of $\text{RREF}(A)$. Similarly, $\text{rank}(B)$ is equal to the number of pivot columns of $\text{RREF}(B)$.

Corollary 1.3.8

Two matrices (with entries in some field) are row equivalent iff they have the same reduced row echelon form.

Proposition 1.6.2

Row equivalent matrices (with entries in some field) have the same rank.

Proof. Fix row equivalent matrices A and B (with entries in some field). By the definition of rank, $\text{rank}(A)$ is equal to the number of pivot columns of A, which is precisely the number of pivot columns of $\text{RREF}(A)$. Similarly, $\text{rank}(B)$ is equal to the number of pivot columns of $\text{RREF}(B)$. But since A and B are row equivalent, Corollary 1.3.8 guarantees that $\text{RREF}(A) = \text{RREF}(B)$. So, $\text{rank}(A) = \text{rank}(B)$. $\square$

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proposition 1.6.3

Let A be an $n \times m$ matrix (with entries in some field $\mathbb{F}$). Then $\text{rank}(A) \leq \min\{n, m\}$.[a]

[a] This means that $\text{rank}(A) \leq n$ (i.e. $\text{rank}(A)$ is at most the number of rows of A) and $\text{rank}(A) \leq m$ (i.e. $\text{rank}(A)$ is at most the number of columns of A).

Proof.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Proof. By definition, $\text{rank}(A)$ is equal to the number of pivot columns of A, and consequently, $\text{rank}(A)$ is at most the number of columns of A, which is m. So, $\text{rank}(A) \leq m$.

On the other hand, $\text{rank}(A)$ is equal to the number of non-zero rows of $\text{RREF}(A)$, and consequently, $\text{rank}(A)$ is at most the number of rows of $\text{RREF}(A)$; since A and $\text{RREF}(A)$ have the same number of rows, we deduce that $\text{rank}(A)$ is at most the number of rows of A, which is n. So, $\text{rank}(A) \leq n$. $\square$

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Let $\mathbb{F}$ be a field, and let A be an $n \times m$ matrix (with entries in some field $\mathbb{F}$). Then $\text{rank}(A) \leq \min\{n, m\}$.[a]

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$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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- **Terminology:** For a field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n \times m}$ (so, A has n rows and m columns):

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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  - if $\text{rank}(A) = n$, then A is said to have *full row rank*;

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Let $\mathbb{F}$ be a field, and let A be an $n \times m$ matrix (with entries in some field $\mathbb{F}$). Then $\text{rank}(A) \leq \min\{n, m\}$.[a]

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$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proposition 1.6.3

Let $\mathbb{F}$ be a field, and let A be an $n \times m$ matrix (with entries in some field $\mathbb{F}$). Then $\text{rank}(A) \leq \min\{n, m\}$.[a]

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[a] This means that $\text{rank}(A) \leq n$ (i.e. $\text{rank}(A)$ is at most the number of rows of A) and $\text{rank}(A) \leq m$ (i.e. $\text{rank}(A)$ is at most the number of columns of A).

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  - if $\text{rank}(A) = n$, then A is said to have *full row rank*;
  - if $\text{rank}(A) = m$, then A is said to have *full column rank*;
  - if $\text{rank}(A) = \min\{n, m\}$, then A is said to have *full rank*;

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proposition 1.6.3

Let $\mathbb{F}$ be a field, and let A be an $n \times m$ matrix (with entries in some field $\mathbb{F}$). Then $\text{rank}(A) \leq \min\{n, m\}$.[a]

[a] This means that $\text{rank}(A) \leq n$ (i.e. $\text{rank}(A)$ is at most the number of rows of A) and $\text{rank}(A) \leq m$ (i.e. $\text{rank}(A)$ is at most the number of columns of A).

- **Terminology:** For a field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n \times m}$ (so, A has n rows and m columns):
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  - if $\text{rank}(A) = \min\{n, m\}$, then A is said to have *full rank*;
  - if $\text{rank}(A) < \min\{n, m\}$, then A is said to be *rank-deficient*.

- As Theorem 1.6.4 (next slide) shows, the number of solutions of a matrix-vector equation $A\mathbf{x} = \mathbf{b}$ can easily be determined if we know the size of the matrix A (i.e. the number of rows and columns of A) and we also know rank(A) and rank($\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$).
  - The detailed proof of Theorem 1.6.4 is in the Lecture Notes.
  - An outline of the proof: on the board!

Theorem 1.6.4

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$ and $\mathbf{b} \in \mathbb{F}^n$. Then

$$\text{rank}(A) \quad \leq \quad \text{rank}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]) \quad \leq \quad \text{rank}(A) + 1.$$

Moreover, all the following hold:

- (a) if $\text{rank}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]) \neq \text{rank}(A)$ (and consequently, $\text{rank}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]) = \text{rank}(A) + 1$), then $A\mathbf{x} = \mathbf{b}$ is inconsistent.
- (b) if $\text{rank}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]) = \text{rank}(A) = m$, then $A\mathbf{x} = \mathbf{b}$ has a unique solution.
- (c) if $\text{rank}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]) = \text{rank}(A) < m$, then $A\mathbf{x} = \mathbf{b}$ has more than one solution, and more precisely,
  - (c.1) if the field $\mathbb{F}$ is finite, then $A\mathbf{x} = \mathbf{b}$ has exactly $|\mathbb{F}|^{m-\text{rank}(A)}$ many solutions,
  - (c.2) if the field $\mathbb{F}$ is infinite, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

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Let $\mathbb{F}$ be a field, and let A be an $n \times m$ matrix (with entries in some field $\mathbb{F}$). Then $\text{rank}(A) \leq \min\{n, m\}$.

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- Our next goal is to prove a derive a couple of corollaries of Theorem 1.6.4 for matrices of full rank.

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- Our next goal is to prove a derive a couple of corollaries of Theorem 1.6.4 for matrices of full rank.
- By definition, a matrix of full rank has full column rank or full row rank (possibly both). We deal with these two cases separately.
- After that, we prove Theorem 1.6.8, which deals with **square** matrices of full rank.
  - Note that such matrices have both full column rank and full row rank.

- In a matrix of full column rank, all columns are pivot columns.

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- So, the reduced row echelon form of such a matrix is of the form

$$\begin{bmatrix} \color{red}{1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \color{red}{1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \color{red}{1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \color{red}{\ddots} & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \color{red}{1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \color{red}{1} \\ \hdashline 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

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$$\left[\begin{array}{cccccc}
1 & 0 & 0 & \dots & 0 & 0 \\
0 & 1 & 0 & \dots & 0 & 0 \\
0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & 0 & \dots & 0 & 1 \\
\hdashline
0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0
\end{array}\right].$$

- More precisely, if we have an $n \times m$ matrix of full column rank, then the reduced row echelon form of that matrix is obtained from the identity matrix I_m by adding $n - m$ many zero rows to the bottom.

- A *homogeneous matrix-vector equation* is a matrix-vector equation of the form $A\mathbf{x} = \mathbf{0}$.

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Corollary 1.6.5

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the following are equivalent:

a. $\text{rank}(A) = m$ (i.e. A has full column rank);
b. the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (i.e. the solution $\mathbf{x} = \mathbf{0}$);
c. there exists some vector $\mathbf{b} \in \mathbb{F}^n$ s.t. the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;
d. for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

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Proof.

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Proof. It is enough to prove following implications:

$$\begin{array}{ccc} \text{(a)} & \Longrightarrow & \text{(d)} \\ \Uparrow & & \Downarrow \\ \text{(c)} & \Longleftarrow & \text{(b)} \end{array}$$

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The implications "(d) $\Longrightarrow$ (b)" and "(b) $\Longrightarrow$ (c)" are obvious. It remains to prove the implications "(c) $\Longrightarrow$ (a)" and "(a) $\Longrightarrow$ (d)."

(a) $\text{rank}(A) = m$ (i.e. A has full column rank);

(c) there exists some vector $\mathbf{b} \in \mathbb{F}^n$ s.t. the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Proof of "(c) $\Longrightarrow$ (a)." Assume that (c) is true, and fix a vector $\mathbf{b} \in \mathbb{F}^n$ s.t. the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. In particular, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, and so Theorem 1.6.4(a) guarantees that $\text{rank}(\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]) = \text{rank}(A)$. Moreover, by Proposition 1.6.3 and Theorem 1.6.4(c), we have that $\text{rank}(A) = m$.[1] Thus, (a) holds.

[1]Indeed, by Proposition 1.6.3, we have that $\text{rank}(A) \leq m$. If $\text{rank}(A) < m$, then Theorem 1.6.4(c) would imply that $A\mathbf{x} = \mathbf{b}$ has more than one solution, a contradiction. So, $\text{rank}(A) = m$.

(a) $\text{rank}(A) = m$ (i.e. A has full column rank);

(d) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

Proof of "(a) $\implies$ (d)." Assume that (a) is true, i.e. that $\text{rank}(A) = m$, and fix a vector $\mathbf{b} \in \mathbb{F}^n$. We must show that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

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$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

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Corollary 1.6.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the following are equivalent:

(a) $\text{rank}(A) = n$ (i.e. A has full row rank);

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the following are equivalent:

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Proof. Suppose first that (a) holds. We must prove (b).

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$$\begin{array}{rcll} n & = & \text{rank}(A) & \text{by (a)} \\ & \leq & \text{rank}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]) & \text{by Theorem 1.6.4} \\ & \leq & n & \text{by Proposition 1.6.3,} \end{array}$$

and it follows that $\text{rank}(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]) = \text{rank}(A) = n$. But now Theorem 1.6.4 guarantees that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent. Thus, (b) holds.

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Proof (continued).

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the following are equivalent:

(a) $\text{rank}(A) = n$ (i.e. A has full row rank);

(b) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Proof (continued). Suppose now that (a) is false; we must show that (b) is false, i.e. that there exists some $\mathbf{b} \in \mathbb{F}^n$ s.t. $A\mathbf{x} = \mathbf{b}$ is inconsistent. Since A is an $n \times m$ matrix and $\text{rank}(A) \neq n$, Proposition 1.6.3 guarantees that $\text{rank}(A) \leq n - 1$. Now, set $U := \text{RREF}(A)$, and let $R_1, \dots, R_k$ be some sequence of elementary row operations that transforms A into U, and for each $i \in \{1, \dots, k\}$, let R_i' be the elementary row operation that reverses (undoes) the elementary row operation R_i. Since U has n rows and $r := \text{rank}(A) \leq n - 1$, we see that the $(r+1)$-th row of U is a zero row. Then the rightmost column of the matrix $\left[\begin{array}{c|c} U & \mathbf{e}_{r+1} \end{array}\right]$ is a pivot column, and consequently, $U\mathbf{x} = \mathbf{e}_{r+1}$ is inconsistent.

Corollary 1.6.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the following are equivalent:

(a) $\text{rank}(A) = n$ (i.e. A has full row rank);

(b) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Proof (continued). Now, we perform the elementary row operations $R'_k, \dots, R'_1$ on the matrix $\left[\begin{array}{c|c} U & \mathbf{e}_{r+1} \end{array}\right]$, and we obtain the matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ for some vector $\mathbf{b} \in \mathbb{F}^n$.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the following are equivalent:

- (a) $\text{rank}(A) = n$ (i.e. A has full row rank);
- (b) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Proof (continued). Now, we perform the elementary row operations $R'_k, \dots, R'_1$ on the matrix $\left[\begin{array}{c|c} U & \mathbf{e}_{r+1} \end{array}\right]$, and we obtain the matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ for some vector $\mathbf{b} \in \mathbb{F}^n$. Since matrices $\left[\begin{array}{c|c} U & \mathbf{e}_{r+1} \end{array}\right]$ and $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ are row equivalent, the matrix-vector equations $U\mathbf{x} = \mathbf{e}_{r+1}$ and $A\mathbf{x} = \mathbf{b}$ are equivalent.

Corollary 1.6.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$. Then the following are equivalent:

(a) $\text{rank}(A) = n$ (i.e. A has full row rank);

(b) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Proof (continued). Now, we perform the elementary row operations $R'_k, \dots, R'_1$ on the matrix $\left[\begin{array}{c:c} U & \mathbf{e}_{r+1} \end{array}\right]$, and we obtain the matrix $\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]$ for some vector $\mathbf{b} \in \mathbb{F}^n$. Since matrices $\left[\begin{array}{c:c} U & \mathbf{e}_{r+1} \end{array}\right]$ and $\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]$ are row equivalent, the matrix-vector equations $U\mathbf{x} = \mathbf{e}_{r+1}$ and $A\mathbf{x} = \mathbf{b}$ are equivalent. Since the matrix-vector equation $U\mathbf{x} = \mathbf{e}_{r+1}$ is inconsistent, it follows that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is also inconsistent. Thus, (b) is false. $\square$

- We now consider the special case of square matrices of full rank.

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Proposition 1.6.7

Let $\mathbb{F}$ be a field. Then for all square matrices $A \in \mathbb{F}^{n \times n}$, we have that $\text{rank}(A) = n$ iff $\text{RREF}(A) = I_n$. In particular, $\text{rank}(I_n) = n$.

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Proof. I_n is a matrix in reduced row echelon form, and it has n pivot columns; so, $\text{rank}(I_n) = n$. Moreover, it is clear that I_n is the **only** reduced row echelon form matrix in $\mathbb{F}^{n \times n}$ of rank n.

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Now, fix any matrix $A \in \mathbb{F}^{n \times n}$. By Proposition 1.6.2, we have that $\text{rank}(A) = \text{rank}\big(\text{RREF}(A)\big)$. Since I_n is the only reduced row echelon form matrix in $\mathbb{F}^{n \times n}$ of rank n, it follows that $\text{rank}(A) = n$ iff $\text{RREF}(A) = I_n$. $\square$

Theorem 1.6.8

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be a **square** matrix. Then the following are equivalent:

(a) $\text{rank}(A) = n$ (i.e. the square matrix A has full rank);

(b) $\text{RREF}(A) = I_n$;

(c) the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (i.e. the solution $\mathbf{x} = \mathbf{0}$);

(d) there exists some vector $\mathbf{b} \in \mathbb{F}^n$ s.t. the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;

(e) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;

(f) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector $A\mathbf{x} = \mathbf{b}$ equation has at most one solution;

(g) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Proof.

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Proof. By Proposition 1.6.7, (a) and (b) are equivalent, and by Corollary 1.6.6, (a) and (g) are equivalent. Further, Corollary 1.6.5 guarantees that (a), (c), (d), and (f) are equivalent. Obviously, (e) implies (f). We complete the proof by showing that (a) implies (e). Assume that (a) holds, and fix a vector $\mathbf{b} \in \mathbb{F}^n$. Since A is a square matrix, (a) guarantees that A has both full column rank and full row rank. Since A has full column rank, Corollary 1.6.5 guarantees that $A\mathbf{x} = \mathbf{b}$ has at most one solution.

Proof. By Proposition 1.6.7, (a) and (b) are equivalent, and by Corollary 1.6.6, (a) and (g) are equivalent. Further, Corollary 1.6.5 guarantees that (a), (c), (d), and (f) are equivalent. Obviously, (e) implies (f). We complete the proof by showing that (a) implies (e). Assume that (a) holds, and fix a vector $\mathbf{b} \in \mathbb{F}^n$. Since A is a square matrix, (a) guarantees that A has both full column rank and full row rank. Since A has full column rank, Corollary 1.6.5 guarantees that $A\mathbf{x} = \mathbf{b}$ has at most one solution. On the other hand, since A has full row rank, Corollary 1.6.6 guarantees that $A\mathbf{x} = \mathbf{b}$ is consistent, i.e. has at least one solution.

Proof. By Proposition 1.6.7, (a) and (b) are equivalent, and by Corollary 1.6.6, (a) and (g) are equivalent. Further, Corollary 1.6.5 guarantees that (a), (c), (d), and (f) are equivalent. Obviously, (e) implies (f). We complete the proof by showing that (a) implies (e). Assume that (a) holds, and fix a vector $\mathbf{b} \in \mathbb{F}^n$. Since A is a square matrix, (a) guarantees that A has both full column rank and full row rank. Since A has full column rank, Corollary 1.6.5 guarantees that $A\mathbf{x} = \mathbf{b}$ has at most one solution. On the other hand, since A has full row rank, Corollary 1.6.6 guarantees that $A\mathbf{x} = \mathbf{b}$ is consistent, i.e. has at least one solution. It now follows that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution, i.e. (e) holds. $\square$

Theorem 1.6.8

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be a **square** matrix. Then the following are equivalent:

(a) $\text{rank}(A) = n$ (i.e. the square matrix A has full rank);
(b) $\text{RREF}(A) = I_n$;
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(e) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;
(f) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has at most one solution;
(g) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent.