Linear Algebra 1

Lecture #1

Systems of linear equations

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This lecture has three parts:

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1. An informal introduction to fields

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2. An introduction to matrices and vectors

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3. Systems of linear equations and row reduction

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- For now, we give a few examples of fields:
  - the field $\mathbb{Q}$ of rational numbers;
  - the field $\mathbb{R}$ of real numbers;
  - the field $\mathbb{C}$ of complex numbers;
  - the field $\mathbb{Z}_p$, where p is a **prime** number.
    - If $n \in \mathbb{N}$ is not prime, then $\mathbb{Z}_n$ is **not** a field.

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- These two operations are commutative and associative, and multiplication is distributive over addition:
  - $a + b = b + a$ and $ab = ba$;
  - $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$;
  - $a(b + c) = ab + ac$.

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  - $a(b + c) = ab + ac$.
- Every field has an "additive identity" 0 and a "multiplicative identity" 1, which satisfy

  $$a + 0 = 0 + a = a \qquad \text{and} \qquad a \cdot 1 = 1 \cdot a = a$$

  for all elements a of the field.

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- For example:
  - the additive inverse of $\sqrt{17}$ in $\mathbb{R}$ is $-\sqrt{17}$, since $\sqrt{17} + (-\sqrt{17}) = 0$ in $\mathbb{R}$.
  - the additive inverse of $2 - i$ in $\mathbb{C}$ is $-2 + i$, since $(2 - i) + (-2 + i) = 0$ in $\mathbb{C}$;
  - the additive inverse of 3 in $\mathbb{Z}_5$ is 2 (and we write $-3 = 2$), since $3 + 2 = 0$ in $\mathbb{Z}_5$;
  - the additive inverse of 4 in $\mathbb{Z}_5$ is 1 (and we write $-4 = 1$), since $4 + 1 = 0$ in $\mathbb{Z}_5$;
  - the additive inverse of 2 in $\mathbb{Z}_3$ is 1 (and we write $-2 = 1$), since $2 + 1 = 0$ in $\mathbb{Z}_3$.

- Every **non-zero** element a of a field has a "multiplicative inverse," denoted by a^{-1}, which is a number we can multiply a by in order to obtain 1.

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- For example:
  - the multiplicative inverse of $\sqrt{17}$ in $\mathbb{R}$ is $\frac{1}{\sqrt{17}}$, because $\sqrt{17} \cdot \frac{1}{\sqrt{17}} = 1$ in $\mathbb{R}$;
  - the multiplicative inverse of $2 - i$ is $\frac{2}{5} + \frac{1}{5}i$, because $(2 - i)(\frac{2}{5} + \frac{1}{5}i) = 1$ in $\mathbb{C}$;
  - the multiplicative inverse of 3 in $\mathbb{Z}_5$ is 2 (and we write $3^{-1} = 2$), since $3 \cdot 2 = 1$ in $\mathbb{Z}_5$;
  - the multiplicative inverse of 4 in $\mathbb{Z}_5$ is 4 (and we write $4^{-1} = 4$), since $4 \cdot 4 = 1$ in $\mathbb{Z}_5$;
  - the multiplicative inverse of 2 in $\mathbb{Z}_3$ is 2 (and we write $2^{-1} = 2$), since $2 \cdot 2 = 1$ in $\mathbb{Z}_3$.

- **Remark:** When working over $\mathbb{Z}_p$ (for a prime number p), it is a good idea to first write out the addition and multiplication tables for $\mathbb{Z}_p$, because this allows us to easily identify additive and multiplicative inverses: for a given $a \in \mathbb{Z}_p$, we simply read off from the tables what number we need to add to a to get zero, and (assuming $a \neq 0$) what number we need to multiply it by to get 1.

- **Remark:** When working over $\mathbb{Z}_p$ (for a prime number p), it is a good idea to first write out the addition and multiplication tables for $\mathbb{Z}_p$, because this allows us to easily identify additive and multiplicative inverses: for a given $a \in \mathbb{Z}_p$, we simply read off from the tables what number we need to add to a to get zero, and (assuming $a \neq 0$) what number we need to multiply it by to get 1.

- **Warning:** The following are **not** fields: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_n$ (where n is a positive integer that is **not** prime).

For the remainder of chapter 1, you may assume that the field $\mathbb{F}$ in question is one of the following: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_p$ (where p is a prime number). However, everything that we prove in this chapter does in fact hold for general fields $\mathbb{F}$, not just the ones listed above.

2. An introduction to matrices and vectors

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 3 & 2 \\ 1 & -1 & -5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

$$2 \times 3 \qquad\qquad 3 \times 2 \qquad\qquad 3 \times 3$$

- A *matrix* is a rectangular array of numbers (typically, elements of some field).
- An $n \times m$ *matrix* (read "n by m matrix") is a matrix with n rows and m columns.

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- A *square matrix* is one that has the same number of rows and columns.
  - So, C is a square matrix, but A and B are not square matrices.
- The *main diagonal* of a square matrix is the diagonal between the upper left corner and the bottom right corner.

$$\begin{bmatrix} \color{red}{3} & 3 & 2 \\ 1 & \color{red}{-1} & -5 \\ -2 & 2 & \color{red}{3} \end{bmatrix}$$

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- So, if $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times m}$, then we have that

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix}.$$

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- **Terminology:** A *real matrix* is a matrix whose entries are real numbers, whereas a *complex matrix* is a matrix whose entries are complex numbers.

$$\mathbf{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -13 \\ 0 \\ 0 \\ \pi \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

- A *column vector*, or simply *vector*, is a matrix with just one column.
- Vectors are typically denoted by bold letters (e.g. $\mathbf{a}$, $\mathbf{u}$, $\mathbf{x}$) or by letters with an arrow on top (e.g. $\vec{a}$, $\vec{u}$, $\vec{x}$).

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- A *non-zero vector* is a vector that has at least one non-zero entry.
- **Notation:** If $\mathbb{F}$ is a field, then the set of all (column) vectors with n entries, all of them in $\mathbb{F}$, is denoted by $\mathbb{F}^n$.
  - Thus, $\mathbb{F}^n = \mathbb{F}^{n \times 1}$.

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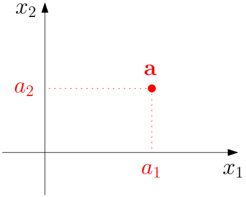
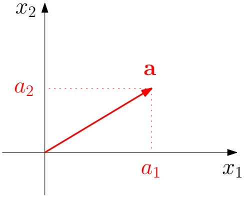
- A vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ in $\mathbb{R}^2$ can be represented in the two-dimensional Euclidean space either as a point or as a line segment with an arrow starting at the origin.





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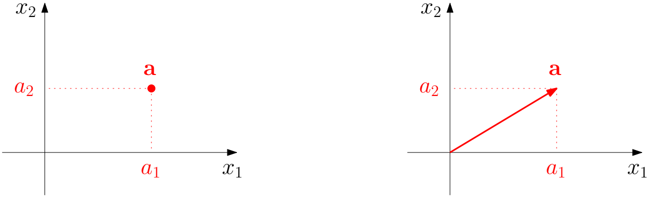

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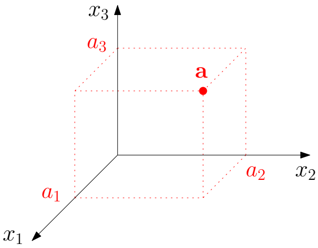

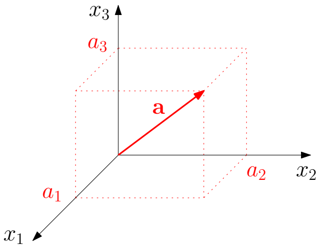

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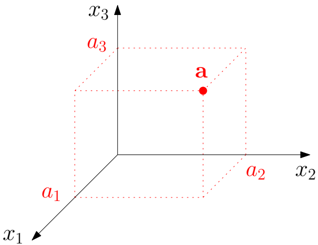

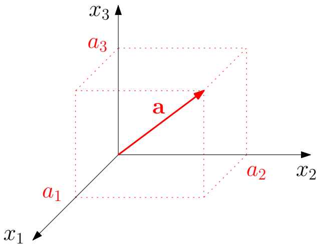

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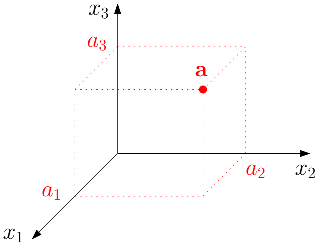

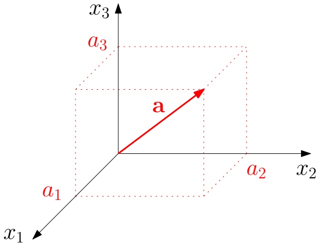

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- Vectors in $\mathbb{R}^n$ for $n \geq 4$ are higher-dimensional analogs of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

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- For example, the following are row vectors:
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  - $\mathbf{c} = \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \end{bmatrix}$.
- The set of all row vectors with n entries, all of them in some field $\mathbb{F}$, is denoted by $\mathbb{F}^{1\times n}$ (i.e. exactly the same way as the set of all $1 \times n$ matrices with entries in $\mathbb{F}$).

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- When we specify a matrix $A \in \mathbb{F}^{n \times m}$ (where $\mathbb{F}$ is some field) in the form

$$A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix},$$

  we mean that $\mathbf{a}_1, \dots, \mathbf{a}_m$ are the columns of A (appearing in that order from left to right in the matrix A), and moreover, $\mathbf{a}_1, \dots, \mathbf{a}_m$ are vectors in $\mathbb{F}^n$.

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- For example, if $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$, where

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

  then $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 4 \end{bmatrix}$.

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we mean that $\mathbf{r}_1, \dots, \mathbf{r}_n$ are the rows of A (appearing in that order from top to bottom in the matrix A), and moreover, $\mathbf{r}_1, \dots, \mathbf{r}_n$ are row vectors in $\mathbb{F}^{1 \times m}$.

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3. Systems of linear equations and row reduction

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- A *linear equation* in the variables $x_1, \ldots, x_m$ is an equation that can be written in the form

$$a_1 x_1 + \cdots + a_m x_m \quad = \quad b,$$

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- For example, $x_1 - 3(x_2 - x_1) = 7x_3 - 4$, with coefficients understood to be in $\mathbb{R}$, is a linear equation because it can be algebraically rearranged to have the form $4x_1 - 3x_2 - 7x_3 = -4$, which is obviously a linear equation.

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- On the other hand, equations $x_1^3 + x_2 = 17$ and $x_1 - \sqrt{x_2} = 5$ are **not** linear because of x_1^3 and $\sqrt{x_2}$.

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- For example, the following is a linear system (here, the coefficients are assumed to be in $\mathbb{R}$):

$$\begin{array}{rcrcrcrcr}
2x_1 & + & 7x_2 & & & - & \pi x_4 & = & -\sqrt{3} \\
 & & -3x_2 & + & 17x_3 & - & 3x_4 & = & 2 \\
x_1 & + & x_2 & - & 2x_3 & + & 7x_4 & = & \frac{11}{2}
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- **Remark:** Typographically, we normally arrange equations in our system so that the terms involving the same variable are below each other (i.e. visually in the same column).
- A *solution* of a linear system in variables $x_1, \ldots, x_m$ is a list $s_1, \ldots, s_m$ of numbers (from the same field as the coefficients of the system) such that each equation becomes a true statement when $s_1, \ldots, s_m$ are substituted for $x_1, \ldots, x_m$, respectively.

Example 1.3.1

Consider the linear system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 9 \\ & & 2x_2 & + & 3x_3 & = & 16 \\ x_1 & + & x_2 & - & x_3 & = & 4 \end{array}$$

with coefficients in $\mathbb{R}$. Then

$$\begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 5 \\ x_3 & = & 2 \end{array}$$

is a solution of the system above.

Example 1.3.2

Consider the linear system

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 0 \\ 2x_1 & + & x_2 & = & 1 \end{array}$$

with coefficients in $\mathbb{Z}_3$. Then

$$\begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 2 \end{array}$$

is a solution of the system above.

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- A linear system may have no solutions, may have a unique solution (i.e. exactly one solution), or may have more than one solution.
- A system that has at least one solution is called *consistent*; a system that has no solutions is said to be *inconsistent*.

- Let us consider the geometry of linear systems with real coefficients.

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- Consider the following system of two linear equations in two variables, with coefficients in $\mathbb{R}$.

$$\begin{array}{ccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & = & b_2 \end{array}$$

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- Let us assume that at least one of the coefficients $a_{1,1}, a_{1,2}$ is non-zero, and similarly, that at least one of the coefficients $a_{2,1}, a_{2,2}$ is non-zero.

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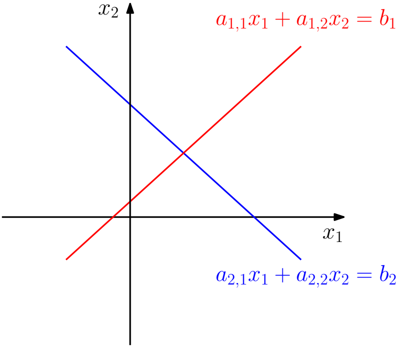
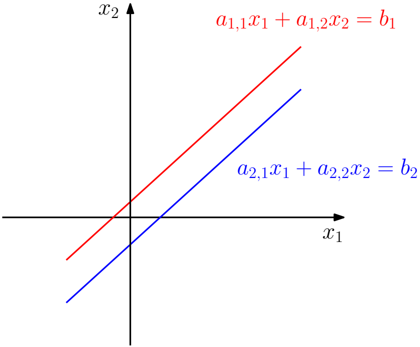
- Then each of the two equations above defines a line in the plane.
- There are three possibilities for these two lines (next three slides):

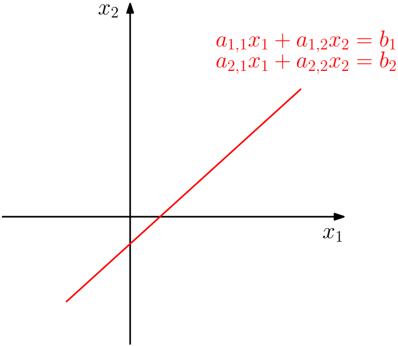
- The two lines may intersect in one point (in this case, the system has a unique solution, and in particular, it is consistent).



- The two lines may be distinct, parallel lines (in this case, the system has no solutions, i.e. it is inconsistent).



- The two lines may be identical (in this case, the system has infinitely many solutions, and in particular, the system is consistent).
  - Note that the two lines may be identical even if the two equations are different. For instance, $x_1 + x_2 = 1$ and $2x_1 + 2x_2 = 2$ define the same line.



- Suppose now that we have a system of two linear equations in three variables (with coefficients in $\mathbb{R}$).

$$\begin{array}{ccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & a_{1,3}x_3 & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & a_{2,3}x_3 & = & b_2 \end{array}$$

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- Let us assume that at least one of the coefficients $a_{1,1}, a_{1,2}, a_{1,3}$ is non-zero, and that at least one of the coefficients $a_{2,1}, a_{2,2}, a_{2,3}$ is non-zero.

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- Suppose we are given a system of n linear equations in m variables, as follows.

$$\begin{array}{ccccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & b_1 \\
a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & b_2 \\
 & & & & & & & \vdots & \\
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\end{array}$$

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- There are two matrices associated with this linear system: the "coefficient matrix" and the "augmented matrix."
- The *coefficient matrix* of this system is the $n \times m$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{bmatrix}.$$

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & b_2 \\ & & & & & & & \vdots & \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \dots & + & a_{n,m}x_m & = & b_n \end{array}$$

- The *augmented matrix* of our linear system is the $n \times (m+1)$ matrix

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc:c} a_{1,1} & a_{1,2} & \dots & a_{1,m} & b_1 \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} & b_n \end{array}\right],$$

where A is the coefficient matrix of the linear system, and

$$\mathbf{b} = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right].$$

- linear system:

$$\begin{array}{ccccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & b_1 \\
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 & & & & & & & \vdots & \\
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\end{array}$$

- augmented matrix:

$$\left[\begin{array}{cccc:c}
a_{1,1} & a_{1,2} & \dots & a_{1,m} & b_1 \\
a_{2,1} & a_{2,2} & \dots & a_{2,m} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n,1} & a_{n,2} & \dots & a_{n,m} & b_n
\end{array}\right]$$

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$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & b_2 \\ & & & & & & & \vdots & \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \dots & + & a_{n,m}x_m & = & b_n \end{array}$$

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- Obviously, a linear system is fully "encoded" by its augmented matrix.

- linear system:

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & b_2 \\ & & & & & & & \vdots & \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \dots & + & a_{n,m}x_m & = & b_n \end{array}$$

- augmented matrix:

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- Obviously, a linear system is fully "encoded" by its augmented matrix.
- The vertical dotted line is optional, but serves as a helpful visual aid.

Example 1.3.3

Find the coefficient matrix and the augmented matrix of the linear system below (with coefficients understood to be in $\mathbb{R}$).

$$\begin{array}{rcrcrcl} 3x_1 & + & 2x_2 & + & 5x_3 & = & 7 \\ & & 3x_2 & - & x_3 & = & 0 \end{array}$$

Solution.

Example 1.3.3

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$$\begin{array}{rcrcrcl} 3x_1 & + & 2x_2 & + & 5x_3 & = & 7 \\ & & 3x_2 & - & x_3 & = & 0 \end{array}$$

Solution. The coefficient matrix of the linear system is

$$\begin{bmatrix} 3 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix},$$

whereas the augmented matrix is

$$\left[\begin{array}{ccr:c} 3 & 2 & 5 & 7 \\ 0 & 3 & -1 & 0 \end{array}\right].$$

$\square$

- Two linear systems (with the same variables) are *equivalent* if they have exactly the same solution set.

- Two linear systems (with the same variables) are *equivalent* if they have exactly the same solution set.
- Now, suppose we are given a system of linear equations such as the one below (with coefficients understood to be in some field $\mathbb{F}$).

$$\begin{array}{ccccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & b_1 \\
a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & b_2 \\
 & & & & & & & \vdots & \\
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- We would like to manipulate this system in a way that allows us to "read off" the solution set of the system.

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 & & & & & & & \vdots & \\
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\end{array}
$$

- We would like to manipulate this system in a way that allows us to "read off" the solution set of the system.
- There are three basic ways that we can manipulate the system in a way that does not change the solution set (i.e. in a way that produces an equivalent linear system).

1. Swap (interchange) two equations.

1. Swap (interchange) two equations.
   - For example, by swapping the first and third equation of the linear system on the left, we obtain the linear system on the right.

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & = & -1 \\ \frac{1}{2}x_1 & & & + & 2x_3 & = & 0 \\ x_1 & + & x_2 & + & 2x_3 & = & 2 \end{array} \quad \longrightarrow \quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 2 \\ \frac{1}{2}x_1 & & & + & 2x_3 & = & 0 \\ x_1 & + & 3x_2 & - & 2x_3 & = & -1 \end{array}$$

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It is obvious that this operation does not alter the solution set.

2. Multiply one equation by a **non-zero** scalar.

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   - For example, by multiplying the second equation of the linear system on the left by 2, we obtain the linear system on the right.

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Let us explain why this does not alter the solution set.

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\end{array}
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Let us explain why this does not alter the solution set. Suppose we have multiplied the i-th equation of our linear system by some scalar $\alpha \neq 0$.

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Let us explain why this does not alter the solution set. Suppose we have multiplied the i-th equation of our linear system by some scalar $\alpha \neq 0$. Obviously, all solutions of the old system are still solutions of the new system. On the other hand, by multiplying the i-th equation of the new system by α^{-1} (the multiplicative inverse of α), we get the old system back.

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   - For example, by multiplying the second equation of the linear system on the left by 2, we obtain the linear system on the right.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 2 \\ \frac{1}{2}x_1 & & & + & 2x_3 & = & 0 \\ x_1 & + & 3x_2 & - & 2x_3 & = & -1 \end{array} \quad \longrightarrow \quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 2 \\ x_1 & & & + & 4x_3 & = & 0 \\ x_1 & + & 3x_2 & - & 2x_3 & = & -1 \end{array}$$

Let us explain why this does not alter the solution set. Suppose we have multiplied the i-th equation of our linear system by some scalar $\alpha \neq 0$. Obviously, all solutions of the old system are still solutions of the new system. On the other hand, by multiplying the i-th equation of the new system by α^{-1} (the multiplicative inverse of α), we get the old system back. So, any solution of the new system is a solution of the old system as well.

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Warning: Do **not** multiply an equation by 0, since that "kills" the equation!

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     - For example, we can swap the first and third row of the matrix on the left to obtain the matrix on the right.

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ \frac{1}{2} & 0 & 2 & 0 \\ 1 & 1 & 2 & 2 \end{array}\right] \quad \overset{R_1 \leftrightarrow R_3}{\sim} \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ \frac{1}{2} & 0 & 2 & 0 \\ 1 & 3 & -2 & -1 \end{array}\right]$$

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$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 2 \\ \frac{1}{2} & 0 & 2 & 0 \\ 1 & 3 & -2 & -1 \end{array}\right] \overset{R_2 \to 2R_2}{\sim} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 2 \\ 1 & 0 & 4 & 0 \\ 1 & 3 & -2 & -1 \end{array}\right]$$

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Note: Instead of "$R_3 \to R_3 + (-1)R_2$," we could also have written (and we typically do write) just "$R_3 \to R_3 - R_2$."

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    - We will, indeed, do this at various points in this course.
  - However, for now, it is useful to think of elementary row operations on matrices as a more compact way of performing the corresponding operations on linear systems.

- **Terminology/Notation:** If one matrix can be obtained from another via some sequence of elementary row operations, then the two matrices are said to be *row equivalent*. If matrices A and B are row equivalent, then we write $A \sim B$.

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- **Remark:** Clearly, if two matrices with at least two columns (and with entries in some field $\mathbb{F}$) are row equivalent, then they encode equivalent linear systems (as augmented matrices).
  - A matrix that only has one column is not the augmented matrix of any linear system.
  - That said, according to our definition, two one-column matrices (i.e. two column vectors) can be row equivalent.

Proposition 1.3.5

Let $\mathbb{F}$ be a field. Then all the following hold:

(a) for all $A \in \mathbb{F}^{n\times m}$, $A \sim A$;

(b) for all $A, B \in \mathbb{F}^{n\times m}$, if $A \sim B$, then $B \sim A$;

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Remark: Proposition 1.3.5 states that, for a field $\mathbb{F}$, row equivalence is an equivalence relation on the set $\mathbb{F}^{n\times m}$.

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Remark: Proposition 1.3.5 states that, for a field $\mathbb{F}$, row equivalence is an equivalence relation on the set $\mathbb{F}^{n\times m}$.

Proof. (a) Fix $A \in \mathbb{F}^{n\times m}$. By, for example, multiplying the first row of A by 1 (i.e. by applying the elementary row operation "$R_1 \to 1R_1$"), we obtain the original matrix A; so, $A \sim A$.

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- *Zero* and *non-zero* columns are defined analogously.
- The *leading entry* of a non-zero row is the leftmost non-zero entry of that row.

- A matrix is in *row echelon form* (or simply *echelon form*), abbreviated *REF*, if it satisfies the following two conditions:
  1. all non-zero rows are above any zero rows;
  2. each leading entry of a non-zero row (other than the top row) is in a column strictly to the right of the column containing the leading entry of the row right above.[1]

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - Here, $\blacksquare$'s represent non-zero numbers, and $*$'s represent arbitrary numbers.

[1] So, all entries in a column below a leading entry of a row are zeros.

- If, in addition, the matrix satisfies the following two conditions, then it is in *reduced row echelon form* (or simply *reduced echelon form*), abbreviated RREF:
   3. the leading entry in each non-zero row is 1;
   4. each leading 1 is the only non-zero entry in its column.

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Here, $*$'s represent arbitrary numbers.

- If, in addition, the matrix satisfies the following two conditions, then it is in *reduced row echelon form* (or simply *reduced echelon form*), abbreviated RREF:
  3. the leading entry in each non-zero row is 1;
  4. each leading 1 is the only non-zero entry in its column.

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - Here, $*$'s represent arbitrary numbers.
- If a matrix is in row echelon form (resp. reduced row echelon form), then we also say that the matrix is a *row echelon matrix* (resp. *reduced row echelon matrix*).

$$
\begin{bmatrix}
0 & \blacksquare & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\qquad
\begin{bmatrix}
0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\
0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\
0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

REF RREF

- A *pivot position* of a matrix in row echelon form is the position of a leading entry of a non-zero row, and a *pivot column* of a matrix in row echelon form is a column that contains a pivot position.
- In our diagram representing a matrix in row echelon form, the pivot positions are the positions of the black squares, and the pivot columns are the columns containing those black squares.
- In the special case of matrices in reduced row echelon form, the pivot positions are the positions of the leading 1's of the non-zero rows, and the pivot columns are the columns containing those leading 1's.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

Theorem 1.3.6

Every matrix (with entries in some field) is row equivalent to a **unique** matrix in reduced row echelon form.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

Theorem 1.3.6

Every matrix (with entries in some field) is row equivalent to a **unique** matrix in reduced row echelon form.

- Proof: Lecture Notes (optional).
  - The proof is an example of a slightly more involved proof by induction.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

Theorem 1.3.6

Every matrix (with entries in some field) is row equivalent to a **unique** matrix in reduced row echelon form.

Corollary 1.3.7

If two row equivalent matrices (with entries in some field) are both in row echelon form, then they have exactly the same pivot positions and exactly the same pivot columns.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF RREF

Theorem 1.3.6

Every matrix (with entries in some field) is row equivalent to a **unique** matrix in reduced row echelon form.

Corollary 1.3.7

If two row equivalent matrices (with entries in some field) are both in row echelon form, then they have exactly the same pivot positions and exactly the same pivot columns.

- Proof of Corollary 1.3.7: Lecture Notes (optional).
  - The proof of Corollary .1.3.7 is not very hard, if we assume that Theorem 1.3.6 is true.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF　　　　　　　　　　RREF

- By Theorem 1.3.6, every matrix A (with entries in some field) is row equivalent to a unique matrix in reduced row echelon form, which we call the *reduced row echelon form* of A, denoted by $\mathrm{RREF}(A)$.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

- By Theorem 1.3.6, every matrix A (with entries in some field) is row equivalent to a unique matrix in reduced row echelon form, which we call the *reduced row echelon form* of A, denoted by $\mathrm{RREF}(A)$.

Corollary 1.3.8

Two matrices (with entries in some field) are row equivalent if and only if they have the same reduced row echelon form.

- Proof: Lecture Notes (optional).
  - The proof of Corollary 1.3.8 is not very hard, is we assume that Theorem 1.3.6 is true.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

- By Theorem 1.3.6, every matrix A (with entries in some field) is row equivalent to a unique matrix in reduced row echelon form, which we call the *reduced row echelon form* of A, denoted by $\mathrm{RREF}(A)$.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF RREF

- By Theorem 1.3.6, every matrix A (with entries in some field) is row equivalent to a unique matrix in reduced row echelon form, which we call the *reduced row echelon form* of A, denoted by RREF(A).
- A *row echelon form* of a matrix A is any matrix that is in row echelon form and is row equivalent to A.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF RREF

- By Theorem 1.3.6, every matrix A (with entries in some field) is row equivalent to a unique matrix in reduced row echelon form, which we call the *reduced row echelon form* of A, denoted by RREF(A).
- A *row echelon form* of a matrix A is any matrix that is in row echelon form and is row equivalent to A.
- A matrix may have more than one row echelon form (i.e. it may be row equivalent to more than one matrix in row echelon form), but by Corollary 1.3.7, all row echelon matrices of a given matrix have the same "shape," i.e. their "black squares" are in the same place.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

- The *pivot positions* and the *pivot columns* of an arbitrary matrix A (with entries from some field) are the pivot positions and the pivot columns, respectively, of any matrix in row echelon form that is row equivalent to A.
  - By Corollary 1.3.7, this is well-defined.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF RREF

- The *pivot positions* and the *pivot columns* of an arbitrary matrix A (with entries from some field) are the pivot positions and the pivot columns, respectively, of any matrix in row echelon form that is row equivalent to A.
  - By Corollary 1.3.7, this is well-defined.
- In particular, if we have computed the reduced row echelon form of a matrix A, then we can immediately identify the pivot positions and the pivot columns of A.

- We now describe an algorithm, called the *row reduction* algorithm, that transforms any matrix (with entries in some field) into a row equivalent matrix that is in reduced row echelon form.
    - This algorithm proves the existence part of Theorem 1.3.6, but not the uniqueness part.

- We now describe an algorithm, called the *row reduction* algorithm, that transforms any matrix (with entries in some field) into a row equivalent matrix that is in reduced row echelon form.
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- The algorithm has two parts: the "forward phase" and the "backward phase."

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  - This algorithm proves the existence part of Theorem 1.3.6, but not the uniqueness part.
- The algorithm has two parts: the "forward phase" and the "backward phase."
- The forward phase transforms the matrix into one in row echelon form.

- We now describe an algorithm, called the *row reduction* algorithm, that transforms any matrix (with entries in some field) into a row equivalent matrix that is in reduced row echelon form.
    - This algorithm proves the existence part of Theorem 1.3.6, but not the uniqueness part.
- The algorithm has two parts: the “forward phase” and the “backward phase.”
- The forward phase transforms the matrix into one in row echelon form.
- The backward phase transforms a matrix in row echelon form into one in **reduced** row echelon form.

- We now describe an algorithm, called the *row reduction* algorithm, that transforms any matrix (with entries in some field) into a row equivalent matrix that is in reduced row echelon form.
  - This algorithm proves the existence part of Theorem 1.3.6, but not the uniqueness part.
- The algorithm has two parts: the "forward phase" and the "backward phase."
- The forward phase transforms the matrix into one in row echelon form.
- The backward phase transforms a matrix in row echelon form into one in **reduced** row echelon form.
- In the description of the algorithm, we will use the word "pivot" to mean the actual number that is in the pivot position in question (or that we intend to move into the pivot position).

- We now describe an algorithm, called the *row reduction* algorithm, that transforms any matrix (with entries in some field) into a row equivalent matrix that is in reduced row echelon form.
    - This algorithm proves the existence part of Theorem 1.3.6, but not the uniqueness part.
- The algorithm has two parts: the "forward phase" and the "backward phase."
- The forward phase transforms the matrix into one in row echelon form.
- The backward phase transforms a matrix in row echelon form into one in **reduced** row echelon form.
- In the description of the algorithm, we will use the word "pivot" to mean the actual number that is in the pivot position in question (or that we intend to move into the pivot position).
- Let's describe the algorithm!

The row reduction algorithm:

- **Forward phase:**
  1. Begin with the leftmost non-zero column. This is a pivot column. The pivot position is at the top of the column.
  2. Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
  3. Use elementary row operations of the form "$R_j \to R_j + \alpha R_i$" (where row i contains the pivot position in question, row j is below row i, and α is a suitable scalar) to create zeros in all positions below the pivot position.
  4. Cover (or ignore) the row containing the pivot position, as well as all the rows (if any) above it. Apply steps 1-4 to the submatrix that remains. Repeat the process until there are no more non-zero rows to modify.
- **Backward phase:**
  5. Beginning with the rightmost pivot column and working upward and to the left, create zeros above each pivot position. If a pivot is not 1, make it 1 by a scaling operation ("$R_i \to \alpha R_i$," for a suitable scalar $\alpha \neq 0$).

Example 1.3.9

Apply the row reduction algorithm to the matrix A below (with entries understood to be in $\mathbb{R}$) in order to compute its reduced row echelon form.

$$A \; := \; \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix}$$

Solution.

Example 1.3.9

Apply the row reduction algorithm to the matrix A below (with entries understood to be in $\mathbb{R}$) in order to compute its reduced row echelon form.

$$A \quad := \quad \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix}$$

Solution. We first implement the forward phase of the algorithm in order to transform the matrix into one in row echelon form, as follows (next slide).

Solution (continued). Forward phase:

$$A = \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix}$$

Solution (continued). Forward phase:

$$A = \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix}$$

$$\overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 0 & -3 & -6 & 3 & 4 & -1 \end{bmatrix}$$

Solution (continued). Forward phase:

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix} \\
&\overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 0 & -3 & -6 & 3 & 4 & -1 \end{bmatrix} \\
&\overset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{cccccc} 2 & 3 & 0 & 11 & -6 & 5 \\ \hdashline 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & -3 & -6 & 3 & 4 & -1 \end{array}\right]
\end{aligned}
$$

Solution (continued). Forward phase:

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix} \\
&\underset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 0 & -3 & -6 & 3 & 4 & -1 \end{bmatrix} \\
&\underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ \hdashline 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & -3 & -6 & 3 & 4 & -1 \end{bmatrix} \\
&\underset{R_3 \to R_3 - \frac{3}{2} R_2}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}
\end{aligned}
$$

Solution (continued). Forward phase:

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix} \\
&\overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 0 & -3 & -6 & 3 & 4 & -1 \end{bmatrix} \\
&\overset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ \hdashline 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & -3 & -6 & 3 & 4 & -1 \end{bmatrix} \\
&\overset{R_3 \to R_3 - \frac{3}{2} R_2}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}
\end{aligned}
$$

The forward phase of the row reduction algorithm is now complete: our matrix is in row echelon form.

Solution (continued). Backward phase:

$$A \sim \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{by the forward phase}$$

Solution (continued). Backward phase:

$$A \sim \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{by the forward phase}$$

$$\overset{\substack{R_1 \to R_1 + 6R_3 \\ R_2 \to R_2 - 2R_3}}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & 0 & 17 \\ 0 & -2 & -4 & 2 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution (continued). Backward phase:

$$A \sim \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{by the forward phase}$$

$$\underset{\substack{R_1 \to R_1 + 6R_3 \\ R_2 \to R_2 - 2R_3}}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & 0 & 17 \\ 0 & -2 & -4 & 2 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R_2 \to -\frac{1}{2} R_2}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & 0 & 17 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution (continued). Backward phase:

$$A \sim \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{by the forward phase}$$

$$\underset{\substack{R_1 \to R_1 + 6R_3 \\ R_2 \to R_2 - 2R_3}}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & 0 & 17 \\ 0 & -2 & -4 & 2 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R_2 \to -\frac{1}{2} R_2}{\sim} \begin{bmatrix} 2 & 3 & 0 & 11 & 0 & 17 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R_1 \to R_1 - 3R_2}{\sim} \begin{bmatrix} 2 & 0 & -6 & 14 & 0 & 8 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution (continued). Backward phase:

$$A \sim \begin{bmatrix} 2 & 3 & 0 & 11 & -6 & 5 \\ 0 & -2 & -4 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{by the forward phase}$$

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$$\overset{R_1 \to \frac{1}{2} R_1}{\sim} \begin{bmatrix} 1 & 0 & -3 & 7 & 0 & 4 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Example 1.3.9

Apply the row reduction algorithm to the matrix A below (with entries understood to be in $\mathbb{R}$) in order to compute its reduced row echelon form.

$$A \;:=\; \begin{bmatrix} 0 & -3 & -6 & 3 & 4 & -1 \\ 2 & 1 & -4 & 13 & -4 & 3 \\ 2 & 3 & 0 & 11 & -6 & 5 \end{bmatrix}$$

Solution (continued). The backward phase of row reduction is now complete: our matrix is in reduced row echelon form. Thus,

$$\mathrm{RREF}(A) \;=\; \begin{bmatrix} 1 & 0 & -3 & 7 & 0 & 4 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

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- There are several other examples in the lecture notes (with matrix entries in $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_5$).

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- We now read off the solution set as follows.

1. If the rightmost column of the augmented matrix (the one to the right of the vertical dotted line) is a pivot column, then the system is inconsistent, i.e. it has no solutions.
   - For example, suppose that by row reduction, we obtained the following matrix (say, with coefficients in $\mathbb{R}$).

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

     This matrix encodes the following linear system:

$$\begin{array}{ccccccc} x_1 & & & - & x_3 & = & 0 \\ & & x_2 & + & 5x_3 & = & 0 \\ & & & & 0 & = & 1 \\ & & & & 0 & = & 0 \end{array}$$

     Because of the equation "$0 = 1$," the system is inconsistent (i.e. it has no solutions).

2. If the rightmost column of the augmented matrix (the one to the right of the vertical dotted line) is **not** a pivot column, but all the other columns **are** pivot columns, then the system has a unique solution.
    - Example: next slide!

- For example, suppose that by row reduction, we obtained the following matrix (say, with coefficients in $\mathbb{R}$).

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix encodes the following linear system:

$$\begin{array}{rcrcrcr} x_1 & & & & & = & -5 \\ & & x_2 & & & = & 0 \\ & & & & x_3 & = & 3 \\ & & & & 0 & = & 0 \end{array}$$

This system is consistent and has a unique solution, which we can immediately read off, as follows.

$$\begin{aligned} x_1 &= -5 \\ x_2 &= 0 \\ x_3 &= 3 \end{aligned}$$

3. If the rightmost column of the augmented matrix (the one to the right of the vertical dotted line) is **not** a pivot column, and at least one of the other columns is also **not** a pivot column, then the system has more than one solution, which we read off as follows.

3. If the rightmost column of the augmented matrix (the one to the right of the vertical dotted line) is **not** a pivot column, and at least one of the other columns is also **not** a pivot column, then the system has more than one solution, which we read off as follows.
    - The variables that correspond to the **non-pivot** columns (we call these variables *free variables*) may take **any** value; these values (called *parameters*) are denoted by letters such as r, s, t.

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- For example, suppose that by row reduction, we obtained the following matrix (say, with coefficients in $\mathbb{R}$).

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This matrix encodes the linear system below.

$$\begin{array}{ccccccccccc} x_1 & + & 2x_2 & & & + & 5x_4 & + & 6x_5 & = & 0 \\ & & & & x_3 & - & x_4 & + & 7x_5 & = & -3 \\ & & & & & & & & 0 & = & 0 \end{array}$$

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$$\begin{aligned} x_1 &= -2r - 5s - 6t \\ x_2 &= r \\ x_3 &= s - 7t - 3 \\ x_4 &= s \\ x_5 &= t \qquad \text{where } r, s, t \in \mathbb{R}. \end{aligned}$$

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- **Remark:** Do not forget to specify which field your parameters come from! Here, we have "$r, s, t \in \mathbb{R}$" because the coefficients of our system are in $\mathbb{R}$.

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    - On the other hand, if our field $\mathbb{F}$ is finite, and our linear system is consistent with exactly k free variables, then the number of solutions of our system is precisely $|\mathbb{F}|^k$ (where $|\mathbb{F}|$ is the cardinality of $\mathbb{F}$, i.e. the number of elements in $\mathbb{F}$).

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    - In particular, if $\mathbb{F} = \mathbb{Z}_p$ for some prime number p, then a consistent system with exactly k free variables has exactly p^k solutions.

- A *homogeneous linear system* is a linear system of the form

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,m}x_m & = & 0 \\ & & & & & & & \vdots & \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \dots & + & a_{n,m}x_m & = & 0 \end{array}$$

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- Some homogeneous linear systems have only the trivial solution, whereas others also have non-trivial solutions.
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- Some homogeneous linear systems have only the trivial solution, whereas others also have non-trivial solutions.
  - This depends on whether there are any free variables.
- When working with homogeneous linear systems, we typically row reduce only the **coefficient matrix**, and not the augmented matrix.

Example 1.3.17

Solve the homogeneous linear system below, with coefficients understood to be in $\mathbb{R}$.

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & & & + & 6x_4 & = & 0 \\ 2x_1 & - & 4x_2 & + & 2x_3 & - & 2x_4 & = & 0 \end{array}$$

How many solutions does this homogeneous linear system have? Does it have any non-trivial solutions?

Solution.

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How many solutions does this homogeneous linear system have? Does it have any non-trivial solutions?

Solution. The **coefficient** matrix of our homogeneous linear system is

$$A := \begin{bmatrix} 2 & -4 & 0 & 6 \\ 2 & -4 & 2 & -2 \end{bmatrix}$$

We row reduce this matrix as follows (next slide):

Solution (continued).

$$
\begin{aligned}
A \quad &= \quad \begin{bmatrix} 2 & -4 & 0 & 6 \\ 2 & -4 & 2 & -2 \end{bmatrix} \\
&\overset{R_2 \to R_2 - R_1}{\sim} \quad \begin{bmatrix} 2 & -4 & 0 & 6 \\ 0 & 0 & 2 & -8 \end{bmatrix} \\
&\overset{\substack{R_1 \to \frac{1}{2} R_1 \\ R_2 \to \frac{1}{2} R_2}}{\sim} \quad \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}.
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Solution (continued).

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\end{aligned}
$$

The last matrix from the calculation above is in reduced row echelon form, and so

$$
\mathrm{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}.
$$

Solution (continued). Reminder:

- $A = \begin{bmatrix} 2 & -4 & 0 & 6 \\ 2 & -4 & 2 & -2 \end{bmatrix}$,
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- We must keep in mind that A is the **coefficient** matrix of our linear system.

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- However, we do not normally write all this! We simply keep track of it mentally.

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We now continue our computation. We see from the matrix $\mathrm{RREF}(A)$ that the pivot columns of the coefficient matrix A are its first and third column. So, x_1, x_3 are the basic variables, and x_2, x_4 are the free variables.

Solution (continued). Reminder: $\text{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$.

We now continue our computation. We see from the matrix RREF(A) that the pivot columns of the coefficient matrix A are its first and third column. So, x_1, x_3 are the basic variables, and x_2, x_4 are the free variables. Further, we see from RREF(A) that our original linear system is equivalent to the linear system below.

$$\begin{array}{ccccccccc} x_1 & - & 2x_2 & & & + & 3x_4 & = & 0 \\ & & & & x_3 & - & 4x_4 & = & 0 \end{array}$$

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We now read off the solutions as follows:

$$\begin{array}{rcl} x_1 & = & 2s - 3t \\ x_2 & = & s \\ x_3 & = & 4t \\ x_4 & = & t \qquad\qquad \text{where } s, t \in \mathbb{R}. \end{array}$$

Solution (continued). Reminder: Our general solution was

$$\begin{aligned} x_1 &= 2s - 3t \\ x_2 &= s \\ x_3 &= 4t \\ x_4 &= t \qquad \text{where } s, t \in \mathbb{R}. \end{aligned}$$

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Since our system has free variables (in fact, two of them), and since we are working over the infinite field $\mathbb{R}$, we see that our system has infinitely many solutions. In particular, our system has a non-trivial solution (in fact, it has infinitely many of them). $\square$

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    - Optional (for the ambitious).