# Linear Algebra 1

## Lecture #0

## Mathematical induction. Modular arithmetic. Arithmetic in $\mathbb{Z}_n$

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**Notation:** Throughout this course, we will use the following notation:

- $\mathbb{N}$ is the set of all natural numbers (positive integers);
- $\mathbb{N}_0$ is the set of all non-negative integers;
- $\mathbb{Z}$ is the set of all integers;
- $\mathbb{Q}$ is the set of all rational numbers;
- $\mathbb{R}$ is the set of all real numbers;
- $\mathbb{C}$ is the set of all complex numbers.

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1. Mathematical induction;
2. Modular arithmetic;
3. Arithmetic in $\mathbb{Z}_n$ and Fermat's Little Theorem

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## 1 Mathematical induction

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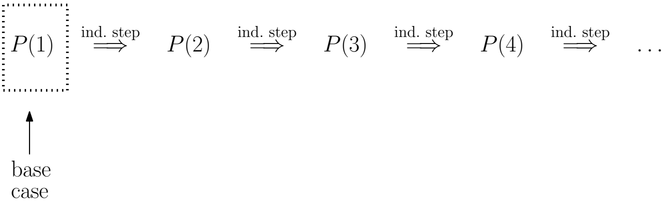
- Let $P(n)$ be a statement about the number $n$. In order to prove that $P(n)$ holds for every positive integer $n$, it suffices to prove the following two statements:
  - **Base case:** $P(1)$ is true;
  - **Induction step:** for every positive integer $n$, if $\underbrace{P(n) \text{ is true}}_{\text{"induction hypothesis"}}$, then $P(n+1)$ is true.



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*Solution.* Let $P(n)$ be the statement that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Thus:

- $P(1)$ is the statement that $1 = \frac{1\cdot(1+1)}{2}$;
- $P(2)$ is the statement that $1 + 2 = \frac{2\cdot(2+1)}{2}$;
- $P(3)$ is the statement that $1 + 2 + 3 = \frac{3\cdot(3+1)}{2}$;
- etc.

We need to prove that the statement $P(n)$ is true for all positive integers $n$.

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**Induction step:** Fix a positive integer $n$, and assume inductively that $P(n)$ is true. We must show that $P(n+1)$ is true.
The induction hypothesis states that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
Using this, we must prove that

$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)\big((n+1)+1\big)}{2}$. We compute (next slide):

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$$\begin{aligned}
1 + 2 + \cdots + n + (n+1) &= (1 + 2 + \cdots + n) + (n+1) \\
&\overset{\substack{\text{ind.}\\\text{hyp.}}}{=} \frac{n(n+1)}{2} + (n+1) \\
&= (n+1)\left(\frac{n}{2} + 1\right) \\
&= \frac{(n+1)\big((n+1)+1\big)}{2}.
\end{aligned}$$

Thus, $P(n+1)$ is true. This completes the induction. $\square$

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$$\boxed{P(k)} \overset{\text{ind. step}}{\Longrightarrow} P(k+1) \overset{\text{ind. step}}{\Longrightarrow} P(k+2) \overset{\text{ind. step}}{\Longrightarrow} P(k+3) \overset{\text{ind. step}}{\Longrightarrow} \dots$$

base case

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**Induction step:** Fix an integer $n \geq 4$, and assume inductively that $3n < 2^n$. We must show that $3(n+1) < 2^{n+1}$. We observe the following:

$$\begin{aligned} 3(n+1) &= 3n+3 \\ &< 2^n+3 && \text{by the induction hypothesis} \\ &< 2^n+2^2 \\ &< 2^n+2^n && \text{because } n > 2 \\ &= 2^{n+1} \end{aligned}$$

Thus, the statement is true for $n+1$. This completes the induction. $\square$

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- In this case, we will have a slightly modified induction step ("$P(n) \Longrightarrow P(n+\ell)$" instead of "$P(n) \Longrightarrow P(n+1)$"), and we will have $\ell$ base cases, namely, $P(k), P(k+1), \ldots, P(k+\ell-1)$.

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- More precisely, we will need to prove the following (next slide):

- **Base case:** $P(k), P(k+1), \ldots, P(k+\ell-1)$ are true;
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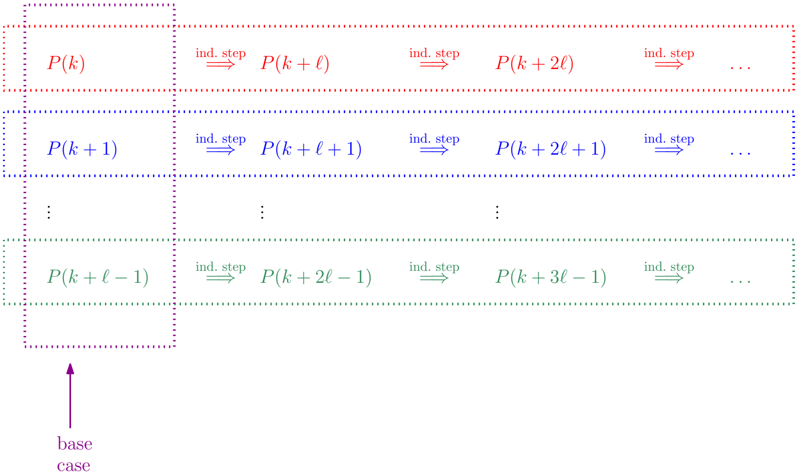

**Example 0.1.3**

Suppose you have an unlimited number of 3 Kč stamps and 5 Kč stamps (and no other stamps). Show that you can pay any amount of postage greater or equal to 8 Kč (as long as it is in whole Kč).

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*Solution.* We need to show that any integer $n \geq 8$ (our postage in Kč) can be expressed in the form

$$n = 3a + 5b,$$

where $a$ and $b$ are non-negative integers (the number of 3 Kč and 5 Kč stamps, respectively, that we can use to pay our $n$ Kč postage).

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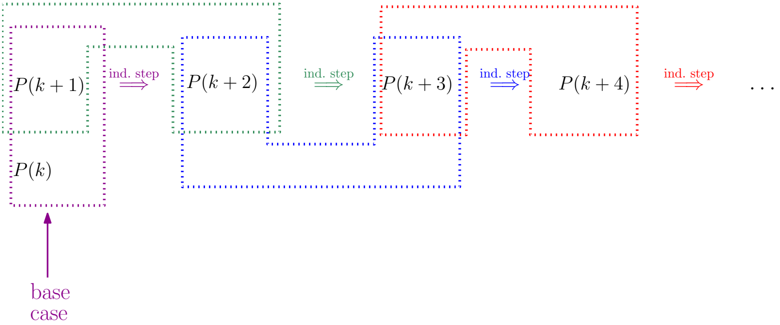
Illustration for $\ell = 2$:



## Example 0.1.4

The *Fibonacci numbers* are defined as follows:

- $F(1) = F(2) = 1$;
- $F(n+2) = F(n) + F(n+1)$ for all positive intgers $n$.

Prove that $F(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$ for all positive integers $n$.

*Solution.*

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*Solution.* The general term is defined in terms of the previous two terms. Thus, instead of one base case, we have two: $n = 1$ and $n = 2$.

**Remark:** If the general term were defined in terms of, say, the previous fifteen terms, then we would have fifteen base cases!

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Prove that $F(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$ for all positive integers $n$.

*Solution (continued).* **Base case:** For $n = 1$, we have:

$$\frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 = F(1).$$

For $n = 2$, we have:

$$\frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} = \frac{(1+2\sqrt{5}+5) - (1-2\sqrt{5}+5)}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1 = F(2).$$

Thus, the statement is true for $n = 1$ and $n = 2$.

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By the induction hypothesis, we have that

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The *Fibonacci numbers* are defined as follows:

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We compute (next slide):

*Solution (continued):*

$$
\begin{aligned}
F(n+2) &\overset{(*)}{=} F(n) + F(n+1) \\
&\overset{(**)}{=} \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}} + \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} \\
&= \frac{4(1+\sqrt{5})^n-4(1-\sqrt{5})^n}{2^{n+2}\sqrt{5}} + \frac{2(1+\sqrt{5})(1+\sqrt{5})^n-2(1-\sqrt{5})(1-\sqrt{5})^n}{2^{n+2}\sqrt{5}} \\
&= \frac{(6+2\sqrt{5})(1+\sqrt{5})^n-(6-2\sqrt{5})(1-\sqrt{5})^n}{2^{n+2}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^2(1+\sqrt{5})^n-(1-\sqrt{5})^2(1-\sqrt{5})^n}{2^{n+2}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n+2}-(1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}},
\end{aligned}
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where (*) follows from the definition of Fibonacci numbers, and (**) follows from the induction hypothesis. This completes the induction.

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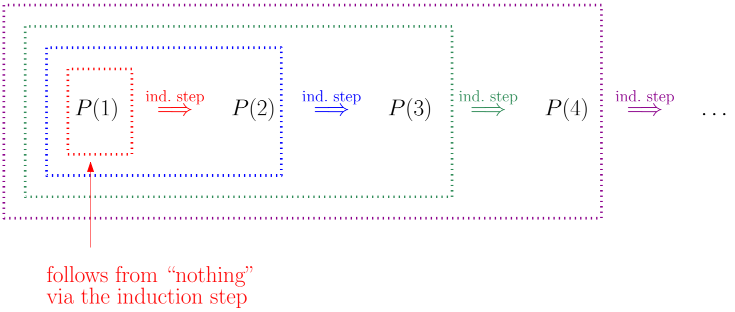

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- Another way of writing the same thing is as follows:
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$$n = \underbrace{n}_{\text{prime}}.$$

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By the induction hypothesis, $n_1$ and $n_2$ can be written as products of primes. Set $n_1 = p_1 \cdot \cdots \cdot p_k$ and $n_2 = q_1 \cdot \cdots \cdot q_\ell$, where $p_1, \ldots, p_k, q_1, \ldots, q_\ell$ are prime numbers.

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Then $n = n_1 n_2 = p_1 \cdot \cdots \cdot p_k \cdot q_1 \cdot \cdots \cdot q_\ell$. Thus, $n$ is a product of primes. This completes the induction. $\square$

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- Given $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, we write $n \mid m$ if $m$ is divisible by $n$, that is, if there exists some $k \in \mathbb{Z}$ s.t. $m = kn$.

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**Example 0.2.1**

- $2 \equiv 17 \pmod{3}$;
- $-13 \equiv 8 \pmod{7}$;
- $-1 \equiv 7 \pmod{4}$;
- $2 \not\equiv 17 \pmod{2}$;
- $-13 \not\equiv 8 \pmod{5}$;
- $-1 \not\equiv 7 \pmod{6}$.

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- For fixed $n \in \mathbb{N}$, every integer is congruent modulo $n$ to exactly one of the following $n$ integers: $0, \ldots, n-1$.
    - As we shall see, doing arithmetic modulo $n$ essentially boils down to doing arithmetic with only $n$ values (namely $0, \ldots, n-1$), as opposed to infinitely many. This is quite useful for certain applications.

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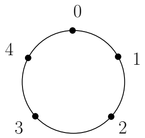
- Congruence modulo $n$ can be visualized in terms of an "$n$-hour clock" (see the picture below for the case $n = 5$).



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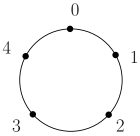


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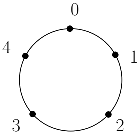


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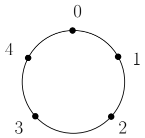


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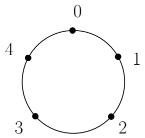


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  - For example, $14 \equiv 4 \pmod{5}$.
- On the other hand, if $a$ is negative, then we make $|a| = -a$ many counterclockwise steps.
  - For example, $-7 \equiv 3 \pmod{5}$.

## Proposition 0.2.2

Let $n \in \mathbb{N}$ and $a, b, c \in \mathbb{Z}$. Then the following hold:

(a) $a \equiv a \pmod{n}$;

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$$a - c \;=\; (a-b) + (b-c) \;=\; kn + \ell n \;=\; (k+\ell)n,$$

i.e. $n \mid (a-c)$. Thus, $a \equiv c \pmod{n}$. $\square$

## Proposition 0.2.3

Let $n \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$, and assume that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then:

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$$(a+c)-(b+d) \quad = \quad (a-b)+(c-d) \quad = \quad kn + \ell n \quad = \quad (k+\ell)n,$$

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The proof of (b) is similar (details: Lecture Notes).

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*Proof (continued).* **Reminder:** $a - b = kn$ and $c - d = \ell n$.

## Proposition 0.2.3

Let $n \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$, and assume that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then:

(a) $a + c \equiv b + d \pmod{n}$;

(b) $a - c \equiv b - d \pmod{n}$;

(c) $ac \equiv bd \pmod{n}$.

*Proof (continued).* **Reminder:** $a - b = kn$ and $c - d = \ell n$.

For (c), we have that

$$\begin{aligned} ac - bd &= ac - ad + ad - bd \\ &= a(c-d) + (a-b)d \\ &= a\ell n + knd \\ &= (a\ell + dk)n, \end{aligned}$$

and so $n \mid (ac - bd)$. Thus, $ac \equiv bd \pmod{n}$. This proves (c). $\square$

**Proposition 0.2.4**

Let $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$. Assume that $a \equiv b \pmod{n}$. Then $a^t \equiv b^t \pmod{n}$ for all integers $t \geq 0$.

*Proof.*

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*Proof.* We proceed by induction on $t$.

**Base case:** $t = 0$. By definition, $r^0 = 1$ for all integers $r$. So, $a^0 = 1 = b^0$, and so $a^0 \equiv b^0 (\mathrm{mod}\ n)$.

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**Induction case:** Fix a non-negative integer $t$, and assume inductively that $a^t \equiv b^t \pmod{n}$. Since we also have that $a \equiv b \pmod{n}$, Proposition 0.2.3(c) implies that $a^t a \equiv b^t b \pmod{n}$, i.e. that $a^{t+1} \equiv b^{t+1} \pmod{n}$. This completes the induction. □

### Proposition 0.2.2

Let $n \in \mathbb{N}$ and $a, b, c \in \mathbb{Z}$. Then the following hold:

(a) $a \equiv a \pmod{n}$;

(b) if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$;

(c) if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

### Proposition 0.2.3

Let $n \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$, and assume that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then:

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- **Notation:** For $a_n, a_{n-1}, \ldots, a_0 \in \{0, 1, \ldots, 9\}$, we define:

$$\overline{a_n a_{n-1} \ldots a_0} \quad := \quad \sum_{k=0}^{n} a_k 10^k.$$

  Thus, $\overline{a_n a_{n-1} \ldots a_0}$ is the number whose first digit is $a_n$, whose second digit is $a_{n-1}$, and so on.

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  - We could eliminate this possibility, but that would result in a messier definition.

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**Proposition 0.2.6**

Let $a = \overline{a_n a_{n-1} \dots a_0}$. Then $a \equiv a_n + a_{n-1} + \cdots + a_0 \pmod 9$. Therefore, a positive integer is divisible by 9 iff the sum of its digits is divisible by 9.

*Proof.*

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*Proof.* By definition, an integer is divisible by 9 iff it is congruent to 0 modulo 9. So, the second statement of the proposition follows immediately from the first.

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$$a \;=\; \overline{a_n a_{n-1} \ldots a_0} \;=\; \sum_{k=0}^{n} a_k 10^k \;\equiv_9\; \sum_{k=0}^{n} a_k \;=\; a_n + a_{n-1} + \cdots + a_0,$$

which is what we needed to show. □

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**Proposition 0.2.7**

Let $a = \overline{a_n a_{n-1} \dots a_0}$. Then $a \equiv a_n + a_{n-1} + \cdots + a_0 \pmod 3$. Therefore, a positive integer is divisible by 3 iff the sum of its digits is divisible by 3.

*Proof.* The proof is completely analogous to that of Proposition 0.2.6: just replace 9 with 3 throughout.

3. Arithmetic in $\mathbb{Z}_n$ and Fermat's Little Theorem

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- For example:
  - $[0]_2 = \{\dots, -4, -2, 0, 2, 4, \dots\}$;
  - $[1]_2 = \{\dots, -3, -1, 1, 3, 5, \dots\}$;
  - $[0]_3 = \{\dots, -6, -3, 0, 3, 6, \dots\}$;
  - $[1]_3 = \{\dots, -5, -2, 1, 4, 7, \dots\}$;
  - $[2]_3 = \{\dots, -4, -1, 2, 5, 8, \dots\}$.

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- We define

$$\mathbb{Z}_n \quad := \quad \{[a]_n \mid a \in \mathbb{Z}\}.$$

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**Proposition 0.2.9**

Let $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$. Then:

(a) if $a \equiv b \pmod{n}$, then $[a]_n = [b]_n$;

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*Proof.*

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*Proof.* This follows from the fact that, by Proposition 0.2.2, congruence modulo $n$ is an equivalence relation on $\mathbb{Z}$. If you are not familiar with the theory of equivalence relations, here is a detailed proof.

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Fix $x \in [a]_n$. Then $x \equiv a \pmod{n}$. Since $a \equiv b \pmod{n}$, Proposition 0.2.2 guarantees that $x \equiv b \pmod{n}$.

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But now by Proposition 0.2.2, we have that $a \equiv b \pmod{n}$. This proves (b). □

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    - $\mathbb{Z} = [0]_n \cup \cdots \cup [n-1]_n$, and
    - the sets $[0]_n, \ldots, [n-1]_n$ are pairwise disjoint.
- If you are familiar with "equivalence relations," then note that congruence modulo $n$ is an equivalence relation on $\mathbb{Z}$ (by Proposition 0.2.2), and the sets $[0]_n, \ldots, [n-1]_n$ are the associated equivalence classes.

- **Reminder:** For a positive integer $n$:
  - $[a]_n := \{x \in \mathbb{Z} \mid x \equiv a \pmod{n}\}$ for all $a \in \mathbb{Z}$;
  - $\mathbb{Z}_n := \{[a]_n \mid a \in \mathbb{Z}\} = \{[0]_n, \ldots, [n-1]_n\}$.
- **Notation:** When working in $\mathbb{Z}_n$, we often write simply $0, \ldots, n-1$ instead of $[0]_n, \ldots, [n-1]_n$, respectively.
  - We may do this **only** if we have previously made it clear that our numbers (which are technically sets of integers) are in $\mathbb{Z}_n$.

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  - We may do this **only** if we have previously made it clear that our numbers (which are technically sets of integers) are in $\mathbb{Z}_n$.

### Example 0.2.10

For $n = 2$, $[0]_2 = \{2t \mid t \in \mathbb{Z}\}$ and $[1]_2 = \{1 + 2t \mid t \in \mathbb{Z}\}$[a], and we have that $\mathbb{Z}_2 = \{[0]_2, [1]_2\}$. Typically, we write simply $\mathbb{Z}_2 = \{0, 1\}$, but technically, 0 stands for the set $[0]_2$, and 1 stands for $[1]_2$.

[a] In other words, $[0]_2$ is the set of all even numbers, and $[1]_2$ is the set of all odd numbers.

### Proposition 0.2.3

Let $n \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$, and assume that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then:

(a) $a + c \equiv b + d \pmod{n}$;

(b) $a - c \equiv b - d \pmod{n}$;

(c) $ac \equiv bd \pmod{n}$.

**Proposition 0.2.3**

Let $n \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$, and assume that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then:

(a) $a + c \equiv b + d \pmod{n}$;

(b) $a - c \equiv b - d \pmod{n}$;

(c) $ac \equiv bd \pmod{n}$.

- By Proposition 0.2.3, for all $n \in \mathbb{N}$ and $a, a', b, b' \in \mathbb{Z}$, if $[a]_n = [a']_n$ and $[b]_n = [b']_n$, then
  - $[a + b]_n = [a' + b']_n$,
  - $[a - b]_n = [a' - b']_n$, and
  - $[ab]_n = [a'b']_n$.

## Proposition 0.2.3

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(a) $a + c \equiv b + d \pmod{n}$;

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- By Proposition 0.2.3, for all $n \in \mathbb{N}$ and $a, a', b, b' \in \mathbb{Z}$, if $[a]_n = [a']_n$ and $[b]_n = [b']_n$, then
  - $[a + b]_n = [a' + b']_n$,
  - $[a - b]_n = [a' - b']_n$, and
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- Thus, we may define addition, subtraction, and multiplication in $\mathbb{Z}_n$ as follows.

**Proposition 0.2.3**

Let $n \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$, and assume that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then:

(a) $a + c \equiv b + d \pmod{n}$;

(b) $a - c \equiv b - d \pmod{n}$;

(c) $ac \equiv bd \pmod{n}$.

- By Proposition 0.2.3, for all $n \in \mathbb{N}$ and $a, a', b, b' \in \mathbb{Z}$, if $[a]_n = [a']_n$ and $[b]_n = [b']_n$, then
  - $[a+b]_n = [a'+b']_n$,
  - $[a-b]_n = [a'-b']_n$, and
  - $[ab]_n = [a'b']_n$.
- Thus, we may define addition, subtraction, and multiplication in $\mathbb{Z}_n$ as follows.
- For $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$, we define
  - $[a]_n + [b]_n = [a+b]_n$;
  - $[a]_n - [b]_n = [a-b]_n$;
  - $[a]_n[b]_n = [ab]_n$.

## Proposition 0.2.11

Let $n \in \mathbb{N}$. Then all the following hold:

(a) addition and multiplication are commutative in $\mathbb{Z}_n$, that is, for all $a, b \in \mathbb{Z}_n$, we have that $a + b = b + a$ and $ab = ba$;

(b) addition and multiplication are associative in $\mathbb{Z}_n$, that is, for all $a, b, c \in \mathbb{Z}_n$, we have that $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$;

(c) multiplication is distributive over addition in $\mathbb{Z}_n$, that is, for all $a, b, c \in \mathbb{Z}_n$, we have that $a(b + c) = ab + ac$.

*Proof.*

## Proposition 0.2.11

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(c) multiplication is distributive over addition in $\mathbb{Z}_n$, that is, for all $a, b, c \in \mathbb{Z}_n$, we have that $a(b + c) = ab + ac$.

*Proof.* This essentially follows from the definition of $\mathbb{Z}_n$, from the fact that addition and multiplication are commutative and associative in $\mathbb{Z}$, and from the fact that multiplication is distributive over addition in $\mathbb{Z}$.

The proof of the commutativity of addition is in the Lecture Notes. The rest is an exercise. □

- Let us now take a look at the addition and multiplication tables for $\mathbb{Z}_n$, for a few small values of $n$.

- Let us now take a look at the addition and multiplication tables for $\mathbb{Z}_n$, for a few small values of $n$.

### Example 0.2.12

Below are the addition and multiplication tables for $\mathbb{Z}_2$.

| $+$ | $[0]_2$ | $[1]_2$ |
|---|---|---|
| $[0]_2$ | $[0]_2$ | $[1]_2$ |
| $[1]_2$ | $[1]_2$ | $[0]_2$ |

| $\cdot$ | $[0]_2$ | $[1]_2$ |
|---|---|---|
| $[0]_2$ | $[0]_2$ | $[0]_2$ |
| $[1]_2$ | $[0]_2$ | $[1]_2$ |

If we omit square brackets and subscripts (as we usually do), we obtain the addition and multiplication tables for $\mathbb{Z}_2$ shown below.

| $+$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

## Example 0.2.13

Below are the addition and multiplication tables for $\mathbb{Z}_3$.[a]

| $+$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

[a] Remember, in this context, 0 stands for $[0]_3$, 1 stands for $[1]_3$, and 2 stands for $[2]_3$.

## Example 0.2.14

Below are the addition and multiplication tables for $\mathbb{Z}_4$.[a]

| $+$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 0 | 2 |
| 3 | 0 | 3 | 2 | 1 |

[a] Remember, in this context, 0 stands for $[0]_4$, 1 stands for $[1]_4$, 2 stands for $[2]_4$, and 3 stands for $[3]_4$.

## Example 0.2.15

Below are the addition and multiplication tables for $\mathbb{Z}_5$.[a]

| $+$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 1 | 3 |
| 3 | 0 | 3 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 2 | 1 |

[a] Remember, in this context, 0 stands for $[0]_5$, 1 stands for $[1]_5$, 2 stands for $[2]_5$, 3 stands for $[3]_5$, and 4 stands for $[4]_5$.

$\mathbb{Z}_2$ :

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| · | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$\mathbb{Z}_3$ :

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| · | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

$\mathbb{Z}_4$ :

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| · | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 0 | 2 |
| 3 | 0 | 3 | 2 | 1 |

$\mathbb{Z}_5$ :

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

| · | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 1 | 3 |
| 3 | 0 | 3 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 2 | 1 |

- **Remark/Notation:** Note that for all positive integers $n$, each number $a$ in $\mathbb{Z}_n$ has a unique "additive inverse," denoted by $-a$, i.e. the number (element of $\mathbb{Z}_n$) that we need to add to $a$ in order to obtain 0 (here, $0 = [0]_n$).

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- When using square brackets and subscripts, we do, of course, get $-[a]_n = [-a]_n = [n-a]_n$ for all positive integers $n$ and all integers $a$.
- However, we will usually work in $\mathbb{Z}_n$ **without** such brackets.
- For small values of $n$, we get the following:
  - in $\mathbb{Z}_2$: $-0 = 0, \quad -1 = 1;$
  - in $\mathbb{Z}_3$: $-0 = 0, \quad -1 = 2, \quad -2 = 1;$
  - in $\mathbb{Z}_4$: $-0 = 0, \quad -1 = 3, \quad -2 = 2, \quad -3 = 1;$
  - in $\mathbb{Z}_5$: $-0 = 0, \quad -1 = 4, \quad -2 = 3, \quad -3 = 2, \quad -4 = 1.$

$\mathbb{Z}_2$ :

| $+$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$\mathbb{Z}_3$ :

| $+$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

$\mathbb{Z}_4$ :

| $+$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 0 | 2 |
| 3 | 0 | 3 | 2 | 1 |

$\mathbb{Z}_5$ :

| $+$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 1 | 3 |
| 3 | 0 | 3 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 2 | 1 |

- **Remark:** Note that for $n = 2, 3, 5$, every non-zero member of $\mathbb{Z}_n$ has a “multiplicative inverse,” i.e. a number that we can multiply it by to get 1.
- However, for $n = 4$, this is not the case.
- As Theorem 0.2.16 and Corollary 0.2.17 (see below) show, this is not an accident!

## Theorem 0.2.16

Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ be relatively prime.[a] Then there exists some $b \in \mathbb{Z}$ s.t. $ab \equiv 1 \pmod{n}$, and therefore, $[a]_n[b]_n = [1]_n$.

[a] This means that the greatest common divisor of $n$ and $a$, denoted by $\gcd(n, a)$, is 1. In other words, the only positive integer that divides both $n$ and $a$ is 1.

*Proof.*

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*Proof.* WTS no two of $0, a, 2a, \ldots, (n-1)a$ are congruent modulo $n$. (Note that this implies that $[a]_n, [2a]_n, \ldots, [(n-1)a]_n$ are pairwise distinct.)

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Suppose otherwise, and fix distinct $i, j \in \{0, \ldots, n-1\}$ s.t. $ia \equiv ja \pmod{n}$. Then $(i-j)a \equiv 0 \pmod{n}$, that is, $n|(i-j)a$.

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Since $n$ and $a$ are relatively prime, it follows that $n|(i-j)$.

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Suppose otherwise, and fix distinct $i, j \in \{0, \dots, n-1\}$ s.t. $ia \equiv ja \pmod{n}$. Then $(i-j)a \equiv 0 \pmod{n}$, that is, $n|(i-j)a$.

Since $n$ and $a$ are relatively prime, it follows that $n|(i-j)$.

But this is impossible because $i, j \in \{0, \dots, n-1\}$ and $i \neq j$, and so $0 < |i-j| < n$.

Thus, no two of $0, a, 2a, \dots, (n-1)a$ are congruent modulo $n$.

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Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ be relatively prime.[a] Then there exists some $b \in \mathbb{Z}$ s.t. $ab \equiv 1 \pmod{n}$, and therefore, $[a]_n[b]_n = [1]_n$.

[a] This means that the greatest common divisor of $n$ and $a$, denoted by $\gcd(n, a)$, is 1. In other words, the only positive integer that divides both $n$ and $a$ is 1.

*Proof (continued).* **Reminder:** No two of $0, a, 2a, \ldots, (n-1)a$ are congruent modulo $n$.

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Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ be relatively prime.[a] Then there exists some $b \in \mathbb{Z}$ s.t. $ab \equiv 1 \pmod{n}$, and therefore, $[a]_n[b]_n = [1]_n$.

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We know that every integer is congruent modulo $n$ to one of the following $n$ integers: $0, 1, 2, \dots, n-1$. We showed above that no two of the following $n$ integers are congruent to each other modulo $n$: $0, a, 2a, \dots, (n-1)a$. It follows that (exactly) one of $0, a, 2a, \dots, (n-1)a$ is congruent to 1 modulo $n$.

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*Proof (continued).* **Reminder:** No two of $0, a, 2a, \ldots, (n-1)a$ are congruent modulo $n$.

We know that every integer is congruent modulo $n$ to one of the following $n$ integers: $0, 1, 2, \ldots, n-1$. We showed above that no two of the following $n$ integers are congruent to each other modulo $n$: $0, a, 2a, \ldots, (n-1)a$. It follows that (exactly) one of $0, a, 2a, \ldots, (n-1)a$ is congruent to 1 modulo $n$.

In other words, for exactly one value of $b \in \{0, 1, 2, \ldots, n-1\}$, we have that $ba \equiv 1 \pmod{n}$. For this $b$, we have that $ab \equiv 1 \pmod{n}$, and therefore, $[a]_n[b]_n = [1]_n$. $\square$

## Theorem 0.2.16

Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ be relatively prime.[a] Then there exists some $b \in \mathbb{Z}$ s.t. $ab \equiv 1 \pmod{n}$, and therefore, $[a]_n[b]_n = [1]_n$.

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## Corollary 0.2.17

Let $p \in \mathbb{N}$ be a prime number. Then:

(a) for all $a \in \mathbb{Z}$ s.t. $a$ is not a multiple of $p$, there exists some $b \in \mathbb{Z}$ s.t. $ab \equiv 1 \pmod{p}$, and consequently, $[a]_p[b]_p = [1]_p$;

(b) for all $a \in \mathbb{Z}_p \setminus \{0\}$, there exists some $b \in \mathbb{Z}_p \setminus \{0\}$ s.t. $ab = 1$.[a]

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*Proof.* We first prove (a). Since $p$ is a prime number, every integer that is not a multiple of $p$ is relatively prime to $p$; (a) now follows from Theorem 0.2.17.

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*Proof (continued).* Statement (b) immediately follows from (a).

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- Corollary 0.2.17(b) states that, for a prime number $p$, every number in $\mathbb{Z}_p \setminus \{0\}$ has a multiplicative inverse.

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- Corollary 0.2.17(b) states that, for a prime number $p$, every number in $\mathbb{Z}_p \setminus \{0\}$ has a multiplicative inverse.
- Fermat's Little Theorem (below) is a strengthening of Corollary 0.2.17 in that it gives an actual formula for this multiplicative inverse.

### Fermat's Little Theorem

If $p \in \mathbb{N}$ is a prime number, and $a \in \mathbb{Z}$ is not a multiple of $p$, then $a^{p-1} \equiv 1 \pmod{p}$.

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- We will prove Fermat's Little Theorem in a bit, but first: how does this give a formula for multiplicative inverses?
- For a positive integer $n$ and for $a \in \mathbb{Z}_n$, we define powers of $a$ recursively, as follows:
  - $a^0 = 1$ (where $1 := [1]_n$);
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  So, for a positive integer $m$, we have the familiar formula

$$a^m = \underbrace{a \cdot \cdots \cdot a}_{m},$$

  where it is understood that the multiplication on the right-hand-side is in $\mathbb{Z}_n$.

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  where it is understood that the multiplication on the right-hand-side is in $\mathbb{Z}_n$.
- With this set-up, we can restate Fermat's Little Theorem in two ways, as follows.

- Old version (to be proven later):

**Fermat's Little Theorem**

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- Suppose that $p$ is a **prime** number and that $a \in \mathbb{Z}_p \setminus \{0\}$.
- By Fermat's Little Theorem, $a^{p-2}$ is a "multiplicative inverse" of $a$, i.e. if we multiply $a$ by $a^{p-2}$ (on either side), we obtain 1.
    - That is: $a \cdot a^{p-2} = a^{p-2} \cdot a = 1$.

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- Moreover, it is easy to see that $a^{p-2}$ is the **only** multiplicative inverse of $a$ in $\mathbb{Z}_p$.
- Indeed, if $b \in \mathbb{Z}_p$ satisfies $ab = 1$, then by multiplying both sides by $a^{p-2}$, we obtain

$$\underbrace{a^{p-2} \cdot a}_{=a^{p-1}=1} b \quad = \quad a^{p-2} \cdot 1,$$

  and consequently, $b = a^{p-2}$.

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If $p \in \mathbb{N}$ is a prime number and $a \in \mathbb{Z}_p \setminus \{0\}$, then $a^{p-1} = 1$.

- So, we can say that $a^{p-2}$ is **the** multiplicative inverse of $a$ (denoted by $a^{-1}$), and we write

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- Note, however, that for small values of the prime number $p$, it is easier to read off the multiplicative inverses of non-zero numbers in $\mathbb{Z}_p$ from the multiplication table for $\mathbb{Z}_p$ than it is to compute the $(p-2)$-th powers of those numbers.

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- By taking a quick look at the multiplication tables for $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_5$, we get the following (next slide):

$\mathbb{Z}_2$ :

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| · | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$\mathbb{Z}_3$ :

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| · | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

$\mathbb{Z}_5$ :

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

| · | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 1 | 3 |
| 3 | 0 | 3 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 2 | 1 |

- in $\mathbb{Z}_2$: $1^{-1} = 1$;
- in $\mathbb{Z}_3$: $1^{-1} = 1, \quad 2^{-1} = 2$;
- in $\mathbb{Z}_5$: $1^{-1} = 1, \quad 2^{-1} = 3, \quad 3^{-1} = 2, \quad 4^{-1} = 4$.

- Proof of Fermat's Little Theorem?

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- So, for a positive integer $n$, we have $n! = 1 \cdot 2 \cdot \dots \cdot n$.

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As in the proof of Theorem 0.2.16, no two of $0, a, 2a, \ldots, (p-1)a$ are congruent modulo $p$. For the sake of completeness, here is a full proof.

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Suppose that some two of $0, a, \ldots, (p-1)a$ are congruent modulo $p$. Fix distinct $i, j \in \{0, 1, \ldots, p-1\}$ s.t. $ia \equiv ja \pmod{p}$. Then $(i-j)a \equiv 0 \pmod{p}$, that is, $p|(i-j)a$.

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Suppose that some two of $0, a, \ldots, (p-1)a$ are congruent modulo $p$. Fix distinct $i, j \in \{0, 1, \ldots, p-1\}$ s.t. $ia \equiv ja \pmod{p}$. Then $(i-j)a \equiv 0 \pmod{p}$, that is, $p|(i-j)a$. Since $p$ is prime and does not divide $a$, we see that $p|(i-j)$.

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If $p \in \mathbb{N}$ is a prime number, and $a \in \mathbb{Z}$ is not a multiple of $p$, then $a^{p-1} \equiv 1 \pmod{p}$.

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Since every integer is congruent to exactly one of $0, 1, \ldots, p-1$ modulo $p$, it follows that there exists some rearrangement (i.e. permutation) $r_1, \ldots, r_{p-1}$ of the sequence $1, \ldots, p-1$ s.t.

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and so $(p-1)!a^{p-1} \equiv (p-1)! \pmod{p}$.

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Since $p$ is prime, we see that $p$ and $(p-1)!$ are relatively prime. It follows that $p \mid (a^{p-1} - 1)$, and consequently, $a^{p-1} \equiv 1 \pmod{p}$, which is what we needed to show. □