Linear Algebra 1: HW#9

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due Friday, January 12, 2024, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. (Of course, you may also type, especially if you know how to use LaTeX.) Other formats will **not** be accepted. Please write your **name** on top of the first page of your HW.

Problem 1 (15 points). *Consider the following set of vectors (matrices) in* $\mathbb{Z}_3^{3\times 2}$*:*

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

(a) Determine whether $\mathcal{A}$ *is a linearly independent set in* $\mathbb{Z}_3^{3\times 2}$*.*

(b) Determine whether $\mathcal{A}$ *is a spanning set of* $\mathbb{Z}_3^{3\times 2}$*.*

(c) Determine whether $\mathcal{A}$ *is a basis of* $\mathbb{Z}_3^{3\times 2}$*.*

> ***Remark/Hint:*** *Two of the three questions can be answered without any computation at all. Which two? (You still need to justify your answer, though!)*

Problem 2 (15 points). *Consider the following set of vectors (polynomials) in* $\mathbb{P}^3_{\mathbb{Z}_2}$*:*

$$\mathcal{B} = \left\{ x^2, \ x^3+1, \ x^3+x^2+x+1, \ x^3+x+1, \ x^2+x \right\}.$$

(a) Determine whether $\mathcal{B}$ *is a linearly independent set in* $\mathbb{P}^3_{\mathbb{Z}_2}$*.*

(b) Determine whether $\mathcal{B}$ *is a spanning set of* $\mathbb{P}^3_{\mathbb{Z}_2}$*.*

(c) *Determine whether* $\mathcal{B}$ *is a basis of* $\mathbb{P}^3_{\mathbb{Z}_2}$.

> ***Remark/Hint:*** *Two of the three questions can be answered without any computation at all. Which two? (You still need to justify your answer, though!)*

Problem 3 (15 points). *Consider the following set of vectors (matrices) in* $\mathbb{R}^{3\times 3}$*:*

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & \pi^5 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & \pi^5 \\ \pi^6 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & \pi^5 \\ \pi^6 & \pi^7 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \pi & \pi^2 \\ \pi^3 & \pi^4 & \pi^5 \\ \pi^6 & \pi^7 & \pi^8 \end{bmatrix} \right\}.$$

(a) *Determine whether* $\mathcal{C}$ *is a linearly independent set in* $\mathbb{R}^{3\times 3}$.

(b) *Determine whether* $\mathcal{C}$ *is a spanning set of* $\mathbb{R}^{3\times 3}$.

(c) *Determine whether* $\mathcal{C}$ *is a basis of* $\mathbb{R}^{3\times 3}$.

> ***Remark/Hint:*** *You shouldn't need to do any row reducing in this problem.*

Problem 4 (30 points). *Consider the following polynomials with coefficients in* $\mathbb{Z}_3$*:*

- $p_1(x) = x^2 + 2$;
- $p_2(x) = 2x^3 + 1$;
- $p_3(x) = x^3 + x^2 + 1$;
- $p_4(x) = 2x^2$.

Find a basis $\mathcal{B}_U$ *of* $U := \mathrm{Span}\big(p_1(x), p_2(x), p_3(x), p_4(x)\big)$*, and find a basis* $\mathcal{B}$ *of* $\mathbb{P}^4_{\mathbb{Z}_3}$ *that extends* $\mathcal{B}_U$ *(i.e. that satisfies* $\mathcal{B}_U \subseteq \mathcal{B}$*). Moreover, for each* $i \in \{1, 2, 3, 4\}$ *such that* $p_i(x) \notin \mathcal{B}$*, express* $p_i(x)$ *as a linear combination of the vectors (polynomials) in* $\mathcal{B}$*.*

> ***Remark:*** *Note that the* $p_i(x)$*'s are all of degree at most* 3*. However,* $\mathcal{B}$ *should be a basis of* $\mathbb{P}^4_{\mathbb{Z}_3}$.

Problem 5 (25 points)**.** *Consider the following matrix and vector, with entries understood to be in* $\mathbb{R}$*:*

$$A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 2 & -6 & 4 & 1 \\ 1 & -3 & 2 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -5 \\ -8 \\ -3 \end{bmatrix}.$$

Show that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ *is consistent (and consequently, its solution set is an affine subset of* $\mathbb{R}^4$*). Find an affine frame and an affine basis of the solution set* S *of* $A\mathbf{x} = \mathbf{b}$*.*