Linear Algebra 1: HW#8

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due Friday, January 5, 2024, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. (Of course, you may also type, especially if you know how to use LaTeX.) Other formats will **not** be accepted. Please write your **name** on top of the first page of your HW.

Problem 1 (20 points)**.** *Give a detailed proof of Corollary 4.3.3 from the Lecture Notes (on page 299).*

> ***Remark:*** *An outline of the proof is given in the Lecture Notes. You are supposed to give the full details.*

Problem 2 (30 points)**.**

(a) Either construct a matrix $A \in \mathbb{R}^{4\times 7}$ such that $Col(A) \cong Nul(A)$ (and prove that your matrix A really does have this property), or prove that no such matrix A exists.

(b) Either construct a matrix $B \in \mathbb{R}^{3\times 4}$ such that $Col(B) \cong Nul(B)$ (and prove that your matrix B really does have this property), or prove that no such matrix B exists.

(c) Either construct a matrix $C \in \mathbb{R}^{3\times 8}$ such that $Col(C) \cong Nul(C)$ (and prove that your matrix C really does have this property), or prove that no such matrix C exists.

Problem 3 (50 points)**.** *Consider the linear function $f : \mathbb{Z}_3^4 \to \mathbb{Z}_3^6$ whose standard matrix is*

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix},$$

and consider the following vectors with entries in $\mathbb{Z}_3$*:*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

Set $U := \mathit{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ *and* $V := \mathit{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.[1]

(a) Compute a basis of $f[U]$.

(b) Compute a basis of $f^{-1}[V]$.

> ***Remark:*** *There are several examples of this type in subsection 4.2.4 of the Lecture Notes. Feel free to use a calculator for any row reduction that you perform, but do keep in mind that on the exam, you might be asked to do problems of this type* ***without*** *a calculator.*

[1]So, U is a subspace of $\mathbb{Z}_3^4$, and V is a subspace of $\mathbb{Z}_3^6$.