Linear Algebra 1: HW#6

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due Friday, December 8, 2023, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. (Of course, you may also type, especially if you know how to use LaTeX.) Other formats will **not** be accepted. Please write your **name** on top of the first page of your HW.

Problem 1 (45 points). *Prove parts (b), (c), and (d) of Proposition 3.1.3 from the Lecture Notes, stated below for convenience.*[1]

> **Proposition 3.1.3 from the Lecture Notes.** *Let V be a vector space over a field $\mathbb{F}$. Then all the following hold:*
>
> *(a) for all $\mathbf{v} \in V$, $0\mathbf{v} = \mathbf{0}$;*
>
> *(b) for all $\alpha \in \mathbb{F}$, $\alpha\mathbf{0} = \mathbf{0}$;*
>
> *(c) for all $\mathbf{v} \in V$ and $\alpha \in \mathbb{F}$, if $\alpha\mathbf{v} = \mathbf{0}$, then $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$;*
>
> *(d) for all $\mathbf{v} \in V$, $(-1)\mathbf{v} = -\mathbf{v}$.*

Problem 2 (30 points). *Let V be a vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Using Theorem 3.1.7 from the Lecture Notes, prove that $U \cap W$ and $U + W$ are both subspaces of V.*[2]

Problem 3 (25 points). *Prove Proposition 3.2.2 from the Lecture Notes, stated below for convenience.*

> **Proposition 3.2.2 from the Lecture Notes.** *Let V be a vector space over a field $\mathbb{F}$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{F} \setminus \{0\}$. Then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent if and only if the set $\{\alpha_1\mathbf{v}_1, \dots, \alpha_k\mathbf{v}_k\}$ is linearly independent.*

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[1] Part (a) was proven in the Lecture Notes. You are asked to prove the three remaining parts.

[2] Reminder: $U + W := \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U,\ \mathbf{w} \in W\}$.