Linear Algebra 1: HW#5

Denys Bulavka & Irena Penev
Winter 2023/2024

due Friday, December 1, 2023, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. (Of course, you may also type, especially if you know how to use LaTeX.) Other formats will **not** be accepted. Please write your **name** on top of the first page of your HW.

Problem 1 (30 points)**.** *Let $n \geq 2$ be an integer. Prove that for any permutation $\pi \in S_n$, we have that*

$$sgn(\pi) \quad = \quad (-1)^{k_{even}},$$

where k_{even} is the number of even cycles in the disjoint cycle decomposition of π.[1]

Problem 2 (20 points)**.** *Prove parts (b) and (d) of Proposition 2.2.4 from the Lecture Notes.*

> ***Hint:*** *Imitate the proofs of parts (a) and (c).*

Problem 3 (50 points)**.** *Let $(G, \cdot)$ be a finite group,*[2] *and let H be a subgroup of G. As usual, for $a, b \in G$, we write ab instead of $a \cdot b$. The identity element of G is denoted by e, and the inverse of an element $a \in G$ is denoted by a^{-1}. For all $a \in G$, define*

$$aH \quad := \quad \{ah \mid h \in H\}.$$

For any finite set X, we denote by $|X|$ the cardinality (i.e. the number of elements) of X.

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[1] An *even cycle* is a cycle that contains an even number of elements, and an *odd cycle* is a cycle that contain an odd number of elements. For example, the permutation (1)(27)(39)(4568) in S_9 has three even cycles (and one odd cycle).

[2] This means that $|G|$ is finite, i.e. G has only finitely many elements.

(a) *[5 points] Prove that for all* $a \in G$*, we have that* $a \in aH$ *(and in particular,* $aH \neq \emptyset$*).*

(b) *[15 points] Prove that for all* $a, b \in H$*, either* $aH = bH$ *or* $aH \cap bH = \emptyset$*.*

(c) *[10 points] Prove that there exist some* $a_1, \ldots, a_k \in G$ *such that*

$$(a_1H, \ldots, a_kH)$$

is a partition of G*, that is, such that* $G = a_1H \cup \cdots \cup a_kH$ *and such that* $a_1H, \ldots, a_kH$ *are pairwise disjoint.*

(d) *[15 points] Prove that for all* $a \in G$*, we have that* $|aH| = |H|$*, that is,* aH *and* H *have the same number of elements.*

> ***Hint:*** *Create a bijection from* H *to* aH *(and prove that it really is a bijection).*

(e) *[5 points] Using the previous parts, prove that* $|H| \mid |G|$*, that is, that* $|G|$ *is divisible by* $|H|$*.*

> ***Remark:*** *In each part, you may rely on the previous parts, even if you did not prove those previous parts (or you proved them incorrectly).*