Linear Algebra 1: HW#4

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due Friday, November 17, 2023, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. (Of course, you may also type, especially if you know how to use LaTeX.) Other formats will **not** be accepted. Please write your **name** on top of the first page of your HW.

Problem 1 (20 points)**.** *Use induction to prove Proposition 1.10.1 from the Lecture Notes. For convenience, this proposition is stated below.*

> ***Proposition 1.10.1*** *Let $\mathbb{F}$ be a field,[1] and let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear function. Then for all vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}^m$ and all scalars $\alpha_1, \dots, \alpha_k \in \mathbb{F}$, we have that*
>
> $$f\Big(\sum_{i=1}^{k} \alpha_i \mathbf{v}_i\Big) \quad = \quad \sum_{i=1}^{k} \alpha_i f(\mathbf{v}_i),$$
>
> *or, written in another way, that*
>
> $$f\Big(\alpha_1 \mathbf{v}_1 + \dots + \alpha_k \mathbf{v}_k\Big) \quad = \quad \alpha_1 f(\mathbf{v}_1) + \dots + \alpha_k f(\mathbf{v}_k).$$

Problem 2 (20 points)**.** *Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Assume that A is invertible and that $AB = I_n$. Prove that $A^{-1} = B$.*

> ***Remark:*** *Here, you are asked to prove the part of Proposition 1.11.3 from the Lecture Notes that was left as an exercise.*

Problem 3 (20 points)**.** *Consider the following two matrices, with entries understood to be in $\mathbb{R}$.*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

For each of these two matrices, either express it as a product of elementary matrices,[2] or explain why this is not possible.

[1] For now, you may assume that $\mathbb{F}$ is $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_p$ (for some prime number p).

[2] Note: There may be more than one correct answer.

Problem 4 (20 points)**.** *Let* $f : \mathbb{R}^2 \to \mathbb{R}^3$ *and* $g : \mathbb{R}^3 \to \mathbb{R}^2$ *be the linear functions whose standard matrices are*

$$A_f = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \qquad \textit{and} \qquad A_g = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix},$$

respectively.

(a) Compute the standard matrix of $f \circ g : \mathbb{R}^3 \to \mathbb{R}^3$*. Determine whether* $f \circ g$ *is an isomorphism, and if so, compute* ~~*its standard matrix*~~ *the standard matrix of* $(f \circ g)^{-1}$*.*

(b) Compute the standard matrix of $g \circ f : \mathbb{R}^2 \to \mathbb{R}^2$*. Determine whether* $g \circ f$ *is an isomorphism, and if so, compute* ~~*its standard matrix*~~ *the standard matrix of* $(g \circ f)^{-1}$*.*

Problem 5 (20 points)**.** *Prove or disprove the following statement: "For all matrices* $A, B \in \mathbb{R}^{2\times 2}$*, if the matrix equation* $XA = B$ *has a* ***unique*** *solution, then that solution is an invertible matrix."*

> ***Remark:*** *First, state clearly whether the statement is true or false. If it is true, then prove it. If it is false, then construct a counterexample (and prove that your counterexample really is a counterexample).*