Linear Algebra 1: HW#3

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due Friday, November 10, 2023, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. (Of course, you may also type, especially if you know how to use LaTeX.) Other formats will **not** be accepted. Please write your **name** on top of the first page of your HW.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand. Moreover, make sure you correctly tell the calculator what kinds of numbers you are using,[1] because otherwise, the calculator may give you a wrong answer.

Problem 1 (20 points)**.** *Let*

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \\ 12 & 8 & 6 \\ 17 & 12 & 9 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the matrix equation $AX = B$. How many solutions does the equation $AX = B$ have?

[1] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Problem 2 (20 points). *Let*

$$A = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}$$

with entries understood to be in $\mathbb{Z}_3$. Solve the matrix equation $XA = B$. How many solutions does the equation $XA = B$ have?

Problem 3 (20 points). *Consider the following vectors, with entries understood to be in $\mathbb{Z}_2$:*

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{c}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Determine if there exists a linear function $f : \mathbb{Z}_2^3 \to \mathbb{Z}_2^4$ that satisfies the property that

$$f(\mathbf{b}_i) = \mathbf{c}_i \quad \textit{for all } i \in \{1, 2, 3, 4\}.$$

If such a linear function f exists, determine whether it is unique, and if it is unique, find a formula for f.

> ***Remark:*** *If the linear function f exists and is unique, then your formula for it should be of the form*
>
> $$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \qquad \textit{for all } x_1, x_2, x_3 \in \mathbb{Z}_2,$$
>
> *where the question marks are replaced with the appropriate values. If the linear function f does not exist, or if it exists but is not unique, then you do not need to find such a formula.*
>
> ***Hint:*** *This problem essentially boils down to solving a matrix equation of the form $XA = B$. Several examples very similar to this problem are given in subsection 1.10.4 of the Lecture Notes.*

Problem 4 (20 points)**.** *Let $\mathbb{F}$ be a field,*[2] *and let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a function. Prove that the following are equivalent:*

(1) f is linear;

(2) for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^m$ and all scalars $\alpha, \beta \in \mathbb{F}$, we have that

$$f(\alpha\mathbf{u} + \beta\mathbf{v}) \quad = \quad \alpha f(\mathbf{u}) + \beta f(\mathbf{v}).$$

Problem 5 (20 points)**.** *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by*

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) \quad = \quad \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$$

*for all $x_1, x_2, x_3 \in \mathbb{R}$. Using either the **definition** of a linear function or the result of Problem 4, prove that f is linear. Then, find the standard matrix of f.*

> ***Hint:*** *This is almost identical to Example 1.10.2(a) from the Lecture Notes.*

[2] For now, you may assume that $\mathbb{F}$ is either $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{Z}_p$ (for some prime p).