Linear Algebra 1: HW#2

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due Friday, October 27, 2023, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. (Of course, you may also type, especially if you know how to use LaTeX.) Other formats will **not** be accepted. Please write your **name** on top of the first page of your HW.

~~**Remark:** All the problems below should be solved using **induction**. **Solutions that do not use induction will receive no credit.**~~

Problem 1 (25 points). *Let $A \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$. Assume that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, and let $\mathbf{x}_0$ be some solution of the equation $A\mathbf{x} = \mathbf{b}$. Prove that the solution set of the equation $A\mathbf{x} = \mathbf{b}$ is precisely the set $S := \{\mathbf{y} + \mathbf{x}_0 \mid \mathbf{y} \in \mathbb{R}^m, A\mathbf{y} = \mathbf{0}\}$.*[1]

> ***Remark#1:*** *Actually, this is true for general fields $\mathbb{F}$, not just for $\mathbb{R}$ (and in fact, the proof is the same for general fields $\mathbb{F}$ as for $\mathbb{R}$). However, for the sake of simplicity, you are only asked to prove this for $\mathbb{R}$.*
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> ***Remark#2/Hint:*** *Essentially, you are asked to prove that, if the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent, and $\mathbf{x}_0$ is an arbitrarily chosen solution of $A\mathbf{x} = \mathbf{b}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by "shifting" the solution set of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$ by $\mathbf{x}_0$. Note that you must prove two things:*
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> 1. *if $\mathbf{y} \in \mathbb{R}^m$ is a solution of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$, then $\mathbf{y} + \mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$;*
> 2. *any solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ can be expressed in the form $\mathbf{y} + \mathbf{x}_0$, for some solution $\mathbf{y}$ of the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$.*

[1] Here, $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$.

Problem 2 (25 points). *Solve the linear system below, where k is some fixed constant. (The coefficients are assumed to be in $\mathbb{R}$.)*

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & + & 2x_4 & = & -4 \\ 2x_1 & + & 5x_2 & & & + & 5x_4 & = & -3 \\ 2x_1 & + & 4x_2 & + & (k^2-5k)x_3 & + & (k+2)x_4 & = & k-11 \end{array}$$

> ***Remark:*** *Your solutions will depend on k. In fact, the **number** of solutions will also depend on k. You will need to figure out for which (if any) k the system has no solutions, for which (if any) it has a unique solution, and for which (if any) it has infinitely many solutions. If the system is consistent, make sure you specify the solutions!*

Problem 3 (25 points). *Let a and b be real constants. For what values of the real constant c is the vector $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$? Make sure you prove that your answer is correct.*

Problem 4 (25 points). *Consider the following vectors in $\mathbb{Z}_3^4$:*

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

For each of the vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$, determine if it can be expressed as a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ (that is, if it belongs to Span$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$), and if so, express it as such a linear combination, and explain whether your answer is unique.

> ***Hint:*** *Imitate the solutions of Examples 1.5.5 and 1.5.6 from the Lecture Notes.*