Linear Algebra 1: HW#1

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Winter 2023/2024

due Friday, October 13, 2023, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. (Of course, you may also type, especially if you know how to use LaTeX.) Other formats will **not** be accepted. Please write your **name** on top of the first page of your HW.

Remark: All the problems below should be solved using **induction**. **Solutions that do not use induction will receive no credit.**

Problem 1 (25 points)**.** *Compute the* ***last two digits*** *of the number* 2035^{2045} *(and prove that your answer is correct).*

Problem 2 (25 points)**.** *Prove that the number* $2^{n+1} + 3^{2n-1}$ *is divisible by* 7 *for all positive integers* n.

Problem 3 (25 points)**.** *Prove that for all real numbers* $a_1, \dots, a_n$ *such that* $0 \leq a_1, \dots, a_n < 1$, *we have that*

$$(1 - a_1)(1 - a_2) \dots (1 - a_n) \quad \geq \quad 1 - a_1 - a_2 - \dots - a_n.$$

Problem 4 (25 points)**.** *Prove that every positive integer has a binary representation. More precisely, prove that for any positive integer* n, *there exists a sequence* $a_0, \dots, a_k$ *of* 0*'s and* 1*'s, with* $a_k = 1$, *such that*

$$n \quad = \quad \sum_{i=0}^{k} a_i \cdot 2^i$$

(so, $\overline{a_k a_{k-1} \dots a_0}$ *is the binary representation of* n*). You may use the fact that for any positive integer* n, *there exists a non-negative integer* ℓ *such that* $2^\ell \leq n < 2^{\ell+1}$.