# Linear Algebra 1: Tutorial 11

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**Problem 4 from HW#9.** *Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and assume that $U$ is non-trivial (i.e. has at least one non-zero vector) and finite-dimensional. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set in $U$, and let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$. Prove that there exists a linear transformation $f : U \to V$ such that $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_k) = \mathbf{v}_k$.*

**Exercise 1.** *Consider the following polynomials in $\mathbb{P}^4_{\mathbb{Z}_3}$:*

- $p_1(x) = x^4 + 2$;
- $p_2(x) = x^3 + x^2$;
- $p_3(x) = x^4 + x^3 + x^2 + 2$;
- $p_4(x) = 2x^4 + x^3 + x^2 + 1$;
- $p_5(x) = 2x + 1$.

*Determine whether there exists a linear function $f : \mathbb{P}^4_{\mathbb{Z}_3} \to \mathbb{Z}^2_3$ such that*

- $f(p_1(x)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;
- $f(p_2(x)) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$;
- $f(p_3(x)) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$;
- $f(p_4(x)) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;
- $f(p_5(x)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**Exercise 2 from Tutorial 10.** *In each of the following cases, determine whether there exists a **linear** function $f$ that satisfies the given properties. You will need to use Problem 4 from HW#9, plus possibly something extra.*

*(a)* $f : \mathbb{R}^4 \to \mathbb{R}^5$ *that satisfies:*

- $f\Big(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$.

*(b)* $f : \mathbb{R}^4 \to \mathbb{R}^5$ *that satisfies:*

- $f\Big(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T\Big) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}^T$.

*(c)* $f : \mathbb{R}^4 \to \mathbb{R}^5$ *that satisfies:*

- $f\Big(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix}^T\Big) = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 1 & 3 & 3 & 3 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$.

*(d)* $f : \mathbb{P}^3_{\mathbb{R}} \to \mathbb{P}^5_{\mathbb{R}}$ *that satisfies:*

- $f(x+1) = x^2$;
- $f(-x+1) = x^4 + 1$;
- $f(x^2) = 3$.

*(e)* $f : \mathbb{P}^3_{\mathbb{R}} \to \mathbb{P}^5_{\mathbb{R}}$ *that satisfies:*

- $f(x+1) = x^2$;
- $f(-x+1) = x^4 + 1$;
- $f(4x) = 2x^4 - 2x^2 + 2$.

*(f)* $f : \mathbb{P}^3_{\mathbb{R}} \to \mathbb{P}^5_{\mathbb{R}}$ *that satisfies:*

- $f(x+1) = x^2$;
- $f(-x+1) = x^4 + 1$;
- $f(4x) = -2x^4 + 2x^2 - 2$.