Linear Algebra 1: Tutorial 10

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Winter 2022/2023

Problem 2 from HW#7. *Let V be a vector space over a field $\mathbb{F}$, and let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly dependent set of vectors.*[1] *Prove that there exists some index $i \in \{1, \dots, k\}$ such that $\mathbf{a}_i$ is a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_{i-1}$.*[2]

> *Hint: You should slightly adapt one direction of the proof of Proposition 1.4(a) from Lecture Notes 7.*

Problem 4 from HW#7. *Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Prove that*

$$dim(U \cap W) + dim(U + W) \quad = \quad dim(U) + dim(W).$$

> ***Remark:*** *By Problem 4 from HW#6, $U \cap W$ is a subspace of V. By Problem 3 from HW#7, $U + W$ is a subspace of V. Since V is finite-dimensional, so are $U \cap W$ and $U + W$ (by Theorem 1.12 from Lecture Notes 7).*
>
> ***Hint:*** *Start with a basis of $U \cap W$ and then extend it in a suitable way, and show that what you obtain is a basis of $U + W$.*

[1] linearly dependent = not linearly independent

[2] It is possible that $i = 1$. Recall that the "empty sum" is equal to the zero vector.

Exercise 1. *Determine which (if any) of the following is a basis of* $\mathbb{P}^5_{\mathbb{R}}$. *(The polynomials are color coded with alternating colors so that it's easier to see which polynomial begins and ends where.)*

(a) $\left\{x^5+4x^4+3x^3+2x+1,\ x^3-3x+7,\ x^2+x+1,\ 1\right\}$

(b) $\left\{x^5+x^4+x^3+x^2+x+1,\ x^4+x^3+x^2+x+1,\ x^3+x^2+x+1,\right.$
$\left. x^2+x+1,\ x+1,\ 1\right\}$

(c) $\left\{x^5-x^4,\ x^5+x^4,\ x^4,\ x^3+x^2,\ x,\ 1\right\}$

Exercise 2. *In each of the following cases, determine whether there exists a **linear** function f that satisfies the given properties. You will need to use Problem 3 from HW#9,*[3] *plus possibly something extra.*

(a) $f:\mathbb{R}^4\to\mathbb{R}^5$ *that satisfies:*

- $f\left(\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T\right)=\begin{bmatrix}1 & 2 & 3 & 4 & 5\end{bmatrix}^T$;
- $f\left(\begin{bmatrix}1 & 2 & 1 & 2\end{bmatrix}^T\right)=\begin{bmatrix}1 & 2 & 1 & 2 & 1\end{bmatrix}^T$;
- $f\left(\begin{bmatrix}0 & 1 & 0 & 1\end{bmatrix}^T\right)=\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T$.

(b) $f:\mathbb{R}^4\to\mathbb{R}^5$ *that satisfies:*

- $f\left(\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T\right)=\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T$;
- $f\left(\begin{bmatrix}1 & 2 & 1 & 2\end{bmatrix}^T\right)=\begin{bmatrix}1 & 2 & 3 & 4 & 5\end{bmatrix}^T$;
- $f\left(\begin{bmatrix}0 & 1 & 0 & 1\end{bmatrix}^T\right)=\begin{bmatrix}0 & 1 & 2 & 3 & 4\end{bmatrix}^T$.

(c) $f:\mathbb{R}^4\to\mathbb{R}^5$ *that satisfies:*

- $f\left(\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T\right)=\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T$;
- $f\left(\begin{bmatrix}1 & 1 & 2 & 2\end{bmatrix}^T\right)=\begin{bmatrix}2 & 2 & 2 & 2 & 2\end{bmatrix}^T$;
- $f\left(\begin{bmatrix}1 & 3 & 3 & 3\end{bmatrix}^T\right)=\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T$.

[3]

Problem 3 from HW#9. *Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is non-trivial (i.e. has at least one non-zero vector) and finite-dimensional. Let $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ be a linearly independent set in U, and let $\mathbf{v}_1,\dots,\mathbf{v}_k\in V$. Prove that there exists a linear transformation $f:U\to V$ such that $f(\mathbf{u}_1)=\mathbf{v}_1,\dots,f(\mathbf{u}_k)=\mathbf{v}_k$.*

(d) $f : \mathbb{P}^3_{\mathbb{R}} \to \mathbb{P}^5_{\mathbb{R}}$ *that satisfies:*

- $f(x+1) = x^2$;
- $f(-x+1) = x^4 + 1$;
- $f(x^2) = 3$.

(e) $f : \mathbb{P}^3_{\mathbb{R}} \to \mathbb{P}^5_{\mathbb{R}}$ *that satisfies:*

- $f(x+1) = x^2$;
- $f(-x+1) = x^4 + 1$;
- $f(4x) = 2x^4 - 2x^2 + 2$.

(f) $f : \mathbb{P}^3_{\mathbb{R}} \to \mathbb{P}^5_{\mathbb{R}}$ *that satisfies:*

- $f(x+1) = x^2$;
- $f(-x+1) = x^4 + 1$;
- $f(4x) = -2x^4 + 2x^2 - 2$.